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# Rapidity evolution of gluon TMD from low to moderate $x$

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Abstract: We study how the rapidity evolution of gluon transverse momentum dependent distribution changes from nonlinear evolution at small $x \ll 1$ to linear evolution at moderate $x \sim 1$.

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## 1 Introduction

A TMD factorization [1–3] generalizes the usual concept of parton density by allowing PDFs to depend on intrinsic transverse momenta in addition to the usual longitudinal momentum fraction variable. These transverse-momentum dependent parton distributions (also called unintegrated parton distributions) are widely used in the analysis of semi-inclusive processes like semi-inclusive deep inelastic scattering (SIDIS) or dijet production in hadron-hadron collisions (for a review, see ref. [3]). However, the analysis of TMD evolution in these cases is mostly restricted to the evolution of quark TMDs, whereas at high collider energies the majority of produced particles will be small-$x$ gluons. In this case one has to understand the transition between non-linear dynamics at small $x$ and presumably linear evolution of gluon TMDs at intermediate $x$.

The study of the transition between the low-$x$ and moderate-$x$ TMDs is complexified by the fact that there are two non-equivalent definitions of gluon TMDs in small-$x$ and "medium $x$" communities. In the small-$x$ literature the Weizsacker-Williams (WW) unintegrated gluon distribution [5] is defined in terms of the matrix element

$$\sum_X \mathrm{tr}\langle p|D^i UU^\dagger(z_\perp)|X\rangle\langle X|D_i UU^\dagger(0_\perp)|p\rangle \tag{1.1}$$

between target states (typically protons). Here tr is a color trace in the fundamental representation, $\sum_X$ denotes the sum over full set of hadronic states and $U_z$ is a Wilson-line operator — infinite gauge link ordered along the light-like line

$$U(z_\perp) = [\infty n + z_\perp, -\infty n + z_\perp], \qquad [x,y] \equiv \mathrm{P}e^{ig\int du\ (x-y)^\mu A_\mu(ux+(1-u)y)} \tag{1.2}$$

and $D^iU(z_\perp) = \partial^i U(z_\perp) - iA^i(\infty n + z_\perp)U(z_\perp) + iA^i(-\infty n + z_\perp)U(z_\perp)$. In the spirit of rapidity factorization, Bjorken $x$ enters this expression as a rapidity cutoff for Wilson-line operators. Roughly speaking, each gluon emitted by Wilson line has rapidity restricted from above by $\ln x_B$.

One can rewrite the above matrix element (up to some trivial factor) in the form

$$\alpha_s \mathcal{D}(x_B, z_\perp) = -\frac{1}{8\pi^2(p\cdot n)x_B}\int du \sum_X \langle p|\tilde{\mathcal{F}}^a_\xi(z_\perp + un)|X\rangle\langle X|\mathcal{F}^{a\xi}(0)|p\rangle \tag{1.3}$$

where

$$\begin{aligned}
\mathcal{F}^a_\xi(z_\perp + un) &\equiv [\infty n + z_\perp, un + z_\perp]^{am} n^\mu g F^m_{\mu\xi}(un + z_\perp)\\
\tilde{\mathcal{F}}^a_\xi(z_\perp + un) &\equiv n^\mu g \tilde{F}^m_{\mu\xi}(un + z_\perp)[un + z_\perp, \infty n + z_\perp]^{ma}
\end{aligned} \tag{1.4}$$

and define the "WW unintegrated gluon distribution"

$$\mathcal{D}(x_B, k_\perp) = \int d^2 z_\perp\ e^{i(k,z)_\perp}\mathcal{D}(x_B, z_\perp) \tag{1.5}$$

(Here $(k, z)_\perp$ denotes the scalar product in 2-dim transverse Euclidean space.) It should be noted that since Wilson lines in eq. (1.1) are renorm-invariant $\alpha_s\mathcal{D}(x_B, k_\perp)$ does not depend on the renormalization scale $\mu$.

On the other hand, at moderate $x_B$ the unintegrated gluon distribution is defined as [6]

$$\begin{aligned}
\mathcal{D}(x_B, k_\perp, \eta) &= \int d^2 z_\perp\ e^{i(k,z)_\perp}\mathcal{D}(x_B, z_\perp, \eta), \qquad\qquad (1.6)\\
\alpha_s\mathcal{D}(x_B, z_\perp, \eta) &= \frac{-x_B^{-1}}{8\pi^2(p\cdot n)}\int du\ e^{-ix_B u(pn)}\sum_X \langle p|\tilde{\mathcal{F}}^a_\xi(z_\perp + un)|X\rangle\langle X|\mathcal{F}^{a\xi}(0)|p\rangle
\end{aligned}$$

where $|p\rangle$ is an unpolarized target with momentum $p$ (typically proton). There are more involved definitions with eq. (1.6) multiplied by some Wilson-line factors [3, 4] following from CSS factorization [7] but we will discuss the "primordial" TMD (1.6). The Bjorken $x$ is now introduced explicitly in the definition of gluon TMD. However, because light-like Wilson lines exhibit rapidity divergencies, we need a separate cutoff $\eta$ (not necessarily equal to $\ln x_B$) for the rapidity of the gluons emitted by Wilson lines. In addition, the matrix elements (1.6) may have double-logarithmic contributions of the type $(\alpha_s\eta\ln x_B)^n$ while the WW distribution (1.3) has only single-log terms $(\alpha_s \ln x_B)^n$ described by the BK evolution [8–11].

In the present paper we study the connection between rapidity evolution of WW TMD (1.3) at low $x_B$ and (1.6) at moderate $x_B \sim 1$. We will assume $k_\perp^2 \geq$ few GeV$^2$ so that we can use perturbative QCD (but otherwise $k_\perp$ is arbitrary and can be of order of $s$ as in the DGLAP evolution). In this kinematic region we will vary Bjorken $x_B$ and look how non-linear evolution at small $x$ transforms into linear evolution at moderate $x_B$. It should be noted that at least at moderate $x_B$ gluon TMDs mix with the quark ones. In this paper we disregard this mixing leaving the calculation of full matrix for future publications. (For the study of quark TMDs in the low-$x$ region see recent preprint [12].)

In addition, we will present the evolution equation for the fragmentation function

$$\mathcal{D}^{\rm f}(\beta_F, k_\perp, \eta) = \int d^2 z_\perp\ e^{-i(k,z)_\perp} \mathcal{D}^{\rm f}(\beta_F, z_\perp, \eta), \tag{1.7}$$

$$\alpha_s \mathcal{D}^{\rm f}(\beta_F, z_\perp, \eta) = \frac{-\beta_F^{-1}}{8\pi^2 (p\cdot n)} \int du\ e^{i\beta_F u(pn)} \sum_X \langle 0|\tilde{\mathcal{F}}^a_\xi(z_\perp + un)|p+X\rangle\langle p+X|\mathcal{F}^{a\xi}(0)|0\rangle$$

where $p$ is the momentum of the registered hadron. It turns out to be free of non-linear terms, at least in the leading log approximation.

It should be emphasized that we consider gluon TMDs with Wilson links going to $+\infty$ in the longitudinal direction relevant for SIDIS [13]. Note that in the leading order SIDIS is determined solely by quark TMDs but beyond that the gluon TMDs should be taken into account, especially for the description of various processes at future EIC collider (see e.g. the report [14]).

It is worth noting that another gluon TMD with links going to $-\infty$ arises in the study of processes with exclusive particle production (like Drell-Yan or Higgs production), see for example the discussion in ref. [15, 16]. We plan to study it in future publications.

The paper is organized as follows. In section 2 we remind the general logic of rapidity factorization and rapidity evolution. In section 3 we derive the evolution equation of gluon TMD in the light-cone (DGLAP) limit. In section 4 we calculate the Lipatov vertex of the gluon production by the $\mathcal{F}^a_i$ operator and the so-called virtual corrections. The final TMD evolution equation for all $x_B$ and transverse momenta is presented in section 5 and in section 6 we discuss the DGLAP, BK and Sudakov limits of our equation. In section 7 we demonstrate that the linearized evolution equation for unintegrated gluon distribution interpolates between BFKL and DGLAP equations. In section 8 we present the evolution equations for fragmentation TMD and section 9 contains conclusions and outlook. The necessary formulas for propagators near the light cone and in the shock-wave background can be found in appendices.

## 2 Rapidity factorization and evolution

In the spirit of high-energy OPE, the rapidity of the gluons is restricted from above by the "rapidity divide" $\eta$ separating the impact factor and the matrix element so the proper definition of $U_x$ is[1]

$$U^\eta_x = \mathrm{Pexp}\Big[ig\int_{-\infty}^{\infty} du\ p_1^\mu A^\eta_\mu(up_1 + x_\perp)\Big],$$
$$A^\eta_\mu(x) = \int \frac{d^4k}{16\pi^4}\ \theta(e^\eta - |\alpha|)e^{-ik\cdot x} A_\mu(k) \tag{2.1}$$

where the Sudakov variable $\alpha$ is defined as usual, $k = \alpha p_1 + \beta p_2 + k_\perp$. We define the light-like vectors $p_1$ and $p_2$ such that $p_1 = n$ and $p_2 = p - \frac{m^2}{s} n$, where $p$ is the momentum of the target

[1] Alternatively, with the leading-log accuracy one can take the Wilson line slightly off the light cone, see ref. [3]. To pave the way for future NLO calculation we prefer the "rigid cutoff" eq. (2.1) which was used for the NLO calculations in the low-$x$ case [17–20].

particle of mass $m$. We use metric $g^{\mu\nu}=(1,-1,-1,-1)$ so $p\cdot q=(\alpha_p\beta_q+\alpha_q\beta_p)\frac{s}{2}-(p,q)_\perp$. For the coordinates we use the notations $x_\bullet\equiv x_\mu p_1^\mu$ and $x_*\equiv x_\mu p_2^\mu$ related to the light-cone coordinates by $x_*=\sqrt{\frac{s}{2}}x_+$ and $x_\bullet=\sqrt{\frac{s}{2}}x_-$. It is convenient to define Fourier transform of the operator $\mathcal{F}_i^a$

$$\mathcal{F}_i^{a\eta}(\beta_B,k_\perp)=\int d^2z_\perp\ e^{-i(k,z)_\perp}\mathcal{F}_i^{a\eta}(\beta_B,z_\perp),$$
$$\mathcal{F}_i^{a\eta}(\beta_B,z_\perp)\equiv\frac{2}{s}\int dz_*\ e^{i\beta_B z_*}\big([\infty,z_*]_z^{am}gF_{\bullet i}^m(z_*,z_\perp)\big)^\eta \tag{2.2}$$

where the index $\eta$ denotes the rapidity cutoff (2.1) for all gluon fields in this operator. Here we introduced the "Bjorken $\beta_B$" to have similar formulas for the DIS and annihilation matrix elements ($\beta_B=x_B$ in DIS and $\beta_B=\beta_F=\frac{1}{z_F}$ for fragmentation functions). Also, hereafter we use the notation $[\infty,z_*]_z\equiv[\infty_*p_1+z_\perp,\frac{2}{s}z_*p_1+z_\perp]$ where $[x,y]$ stands for the straight-line gauge link connecting points $x$ and $y$ as defined in eq. (1.2). Our convention is that the Latin Lorentz indices always correspond to transverse coordinates while Greek Lorentz indices are four-dimensional.

Similarly, we define

$$\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,k_\perp)=\int d^2z_\perp\ e^{i(k,z)_\perp}\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,z_\perp),$$
$$\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,z_\perp)\equiv\frac{2}{s}\int dz_*\ e^{-i\beta_B z_*}g\big(\tilde{F}_{\bullet i}^m(z_*,z_\perp)[z_*,\infty]_z^{ma}\big)^\eta \tag{2.3}$$

in the complex-conjugate part of the amplitude.

In this notations the unintegrated gluon TMD (1.6) can be represented as

$$\begin{aligned}\langle p|\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,z_\perp)\mathcal{F}^{ai\eta}(\beta_B,0_\perp)|p+\xi p_2\rangle &\equiv\sum_X\langle p|\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,z_\perp)|X\rangle\langle X|\mathcal{F}^{ai\eta}(\beta_B,0_\perp)|p+\xi p_2\rangle\\ &=-4\pi^2\delta(\xi)\beta_B g^2\mathcal{D}(\beta_B,z_\perp,\eta)\end{aligned}\tag{2.4}$$

Hereafter we use a short-hand notation

$$\langle p|\tilde{\mathcal{O}}_1\dots\tilde{\mathcal{O}}_m\mathcal{O}_1\dots\mathcal{O}_n|p'\rangle\equiv\sum_X\langle p|\tilde{T}\{\tilde{\mathcal{O}}_1\dots\tilde{\mathcal{O}}_m\}|X\rangle\langle X|T\{\mathcal{O}_1\dots\mathcal{O}_n\}|p'\rangle \tag{2.5}$$

where tilde on the operators in the l.h.s. of this formula stands as a reminder that they should be inverse time ordered as indicated by inverse-time ordering $\tilde{T}$ in the r.h.s. of the above equation.

As discussed e.g. in ref. [21–23], such martix element can be represented by a double functional integral

$$\begin{aligned}&\langle\tilde{\mathcal{O}}_1\dots\tilde{\mathcal{O}}_m\mathcal{O}_1\dots\mathcal{O}_n\rangle\\ &=\int D\tilde{A}D\tilde{\bar{\psi}}D\tilde{\psi}\ e^{-iS_{\rm QCD}(\tilde{A},\tilde{\psi})}\int DAD\bar{\psi}D\psi\ e^{iS_{\rm QCD}(A,\psi)}\tilde{\mathcal{O}}_1\dots\tilde{\mathcal{O}}_m\mathcal{O}_1\dots\mathcal{O}_n\end{aligned}\tag{2.6}$$

with the boundary condition $\tilde{A}(\vec{x},t=\infty)=A(\vec{x},t=\infty)$ (and similarly for quark fields) reflecting the sum over all intermediate states $X$. Due to this condition, the matrix element (2.4) can be made gauge-invariant by connecting the endpoints of Wilson lines at infinity with the gauge link[2]

$$\begin{aligned}&\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}^{ai}(\beta'_B,y_\perp)|p'\rangle\\ \to\ &\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)[x_\perp+\infty p_1,y_\perp+\infty p_1]\mathcal{F}^{ai}(\beta'_B,y_\perp)|p'\rangle\end{aligned}\tag{2.7}$$

This gauge link is important if we use the light-like gauge $p_1^\mu A_\mu=0$ for calculations [24], but in all other gauges it can be neglected. We will not write it down explicitly but will always assume it in our formulas.

We will study the rapidity evolution of the operator

$$\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,x_\perp)\mathcal{F}_j^{a\eta}(\beta_B,y_\perp)\tag{2.8}$$

Matrix elements of this operator between unpolarized hadrons can be parametrized as [6]

$$\begin{aligned}&\int d^2z_\perp\ e^{i(k,z)_\perp}\langle p|\tilde{\mathcal{F}}_i^{a\eta}(\beta_B,z_\perp)\mathcal{F}_j^{a\eta}(\beta_B,0_\perp)|p+\xi p_2\rangle=2\pi^2\delta(\xi)\beta_Bg^2\mathcal{R}_{ij}(\beta_B,k_\perp;\eta)\\ &\mathcal{R}_{ij}(\beta_B,k_\perp;\eta)=-g_{ij}\mathcal{D}(\beta_B,k_\perp,\eta)+\left(\frac{2k_ik_j}{m^2}+g_{ij}\frac{k_\perp^2}{m^2}\right)\mathcal{H}(\beta_B,k_\perp,\eta)\end{aligned}\tag{2.9}$$

where $m$ is the mass of the target hadron (typically proton). The reason we study the evolution of the operator (2.8) with non-convoluted indices $i$ and $j$ is that, as we shall see below, the rapidity evolution mixes functions $\mathcal{D}$ and $\mathcal{H}$. It should be also noted that our final equation for the evolution of the operator (2.8) is applicable for polarized targets as well.

We shall also study the evolution of fragmentation functions defined by "fragmentation matrix elements" (1.7) of the operator (2.8). If the polarization of the fragmentation hadron is not registered, this matrix element can be parametrized similarly to eq. (2.9) (cf. ref. [6])

$$\begin{aligned}&\int d^2z_\perp\ e^{-i(k,z)_\perp}\sum_X\langle 0|\tilde{\mathcal{F}}_i^{a\eta}(-\beta_F,z_\perp)|p+X\rangle\langle p+\xi p_2+X|\mathcal{F}_j^{a\eta}(-\beta_F,0_\perp)|0\rangle\\ &=2\pi^2\delta(\xi)\beta_Fg^2\left[-g_{ij}\mathcal{D}^{\rm f}(\beta_F,k_\perp,\eta)+\left(\frac{2k_ik_j}{m^2}+g_{ij}\frac{k_\perp^2}{m^2}\right)\mathcal{H}^{\rm f}(\beta_F,k_\perp,\eta)\right]\end{aligned}\tag{2.10}$$

Note that $\beta_F$ should be greater than 1 in this equation, otherwise the cross section vanishes. As to matrix element (2.4), it can be defined with either sign of $\beta_B$ but the deep inelastic scattering corresponds to $\beta_B=x_B>0$. In our calculations we will consider $\beta_B>0$ for simplicity and perform the trivial analytic continuation to negative $\beta_B$ in the final formula (5.2).

In the spirit of rapidity factorization, in order to find the evolution of the operator (2.8) with respect to rapidity cutoff $\eta$ (see eq. (2.1)) one should integrate in the matrix element (2.4) over gluons and quarks with rapidities $\eta>Y>\eta'$ and temporarily "freeze" fields with $Y<\eta'$ to be integrated over later. (For a review, see refs. [25, 26].) In this case, we obtain functional integral of eq. (2.6) type over fields with $\eta>Y>\eta'$ in the "external" fields with $Y<\eta'$. In terms of Sudakov variables we integrate over gluons with $\alpha$ between $\sigma=e^\eta$ and $\sigma'=e^{\eta'}$ and, in the leading order, only the diagrams with gluon emissions are relevant — the quark diagrams will enter as loops at the next-to-leading (NLO) level.

[2]Similarly, this gauge link is implied in eq. (1.1) which is eq. (2.7) at $\beta_B=0$.

To make connections with parton model we will have in mind the frame where target's velocity is large and call the small $\alpha$ fields by the name "fast fields" and large $\alpha$ fields by "slow" fields. Of course, "fast" *vs* "slow" depends on frame but we will stick to naming fields as they appear in the projectile's frame. (Note that in ref. [8, 9] the terminology is opposite, as appears in the target's frame). As discussed in ref. [8, 9], the interaction of "slow" gluons of large $\alpha$ with "fast" fields of small $\alpha$ is described by eikonal gauge factors and the integration over slow fields results in Feynman diagrams in the background of fast fields which form a thin shock wave due to Lorentz contraction. However, in ref. [8, 9] (as well as in all small-$x$ literature) it was assumed that the characteristic transverse momenta of fast and slow fields are of the same order of magnitude. For our present purposes we need to relax this condition and consider cases where the transverse momenta of fast and slow fields do differ. In this case, we need to rethink the shock-wave approach.

Let us figure out how the relative longitudinal size of fast and slow fields depends on their transverse momenta. The typical longitudinal size of fast fields is $\sigma_* \sim \frac{\sigma' s}{l_\perp^2}$ where $l_\perp$ is the characteristic scale of transverse momenta of fast fields. The typical distances traveled by slow gluons are $\sim \frac{\sigma s}{k_\perp^2}$ where $k_\perp$ is the characteristic scale of transverse momenta of slow fields. Effectively, the large-$\alpha$ gluons propagate in the external field of the small-$\alpha$ shock wave, except the case $l_\perp^2 \ll k_\perp^2$ which should be treated separately since the "shock wave" is not necessarily thin in this case. Fortunately, when $l_\perp^2 \ll k_\perp^2$ one can use the light-cone expansion of slow fields and leave at the leading order only the light-ray operators of the leading twist. We will use the combination of shock-wave and light-cone expansions and write the interpolating formulas which describe the leading-order contributions in both cases.

## 3 Evolution kernel in the light-cone limit

As we discussed above, we will obtain the evolution kernel in two separate cases: the "shock wave" case when the characteristic transverse momenta of the background gluon (or quark) fields $l_\perp$ are of the order of typical momentum of emitted gluon $k_\perp$ and the "light cone" case when $l_\perp^2 \ll k_\perp^2$. It is convenient to start with the light-cone situation and consider the one-loop evolution of the operator $\tilde{\mathcal{F}}_i^{a\eta}(\beta_B, x_\perp)\mathcal{F}_j^{a\eta}(\beta_B, y_\perp)$ in the case when the background fields are soft so we can use the expansion of propagators in external fields near the light cone [27, 28].

In the leading order there is only one "quantum" gluon and we get the typical diagrams of figure 1 type. One sees that the evolution kernel consist of two parts: "real" part with the emission of a real gluon and a "virtual" part without such emission. The "real" production part of the kernel can be obtained as a square of a Lipatov vertex — the amplitude of the emission of a real gluon by the Wilson-line operator $\mathcal{F}_i^a$:

$$
\begin{aligned}
&\langle \tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)\rangle^{\ln\sigma} \\
&= -\int_{\sigma'}^{\sigma} \frac{d\!\!\bar{}\,\alpha}{2\alpha} d\!\!\bar{}^2 k_\perp \big(\langle \lim_{k^2\to 0} k^2 \tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\tilde{A}_\rho^m(k)\rangle\langle \lim_{k^2\to 0} k^2 A^{m\rho}(k)\mathcal{F}_j^a(\beta_B, y_\perp)\rangle\big)^{\ln\sigma'} \qquad (3.1)
\end{aligned}
$$

Hereafter we use the space-saving notation $d\!\!\bar{}^n p \equiv \frac{d^n p}{(2\pi)^n}$.

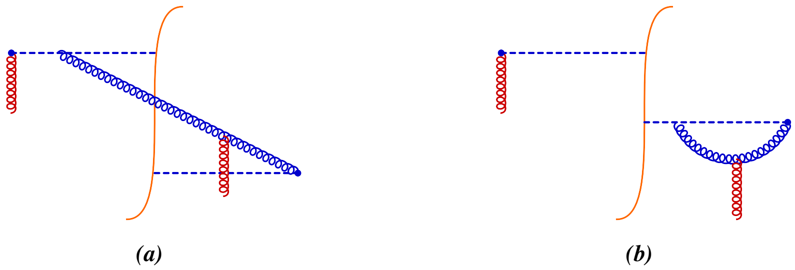


**Figure 1**. Typical diagrams for production (a) and virtual (b) contributions to the evolution kernel. The dashed lines denote gauge links.

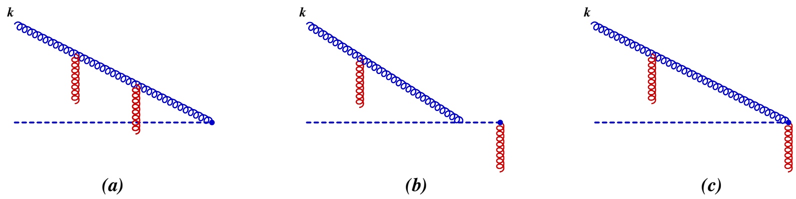


**Figure 2**. Lipatov vertex of gluon emission.

### 3.1 Lipatov vertex

As we mentioned, the production ("real") part of the kernel corresponds to square of Lipatov vertex describing the emission of a gluon by the operator $\mathcal{F}_i^a$. The Lipatov vertex is defined as

$$L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2\to 0} k^2 \langle T\{A^a_\mu(k)\mathcal{F}^b_i(\beta_B, y_\perp)\}\rangle \tag{3.2}$$

(To simplify our notations, we will often omit label $\eta$ for the rapidity cutoff (2.1) but it will be always assumed when not displayed).

We will use the background-Feynman gauge. The three corresponding diagrams are shown in figure 2.

#### 3.1.1 Emission of soft gluon near the light cone

In accordance with general background-field formalism we separate the gluon field into the "classical" background part and "quantum" part

$$A_\mu \;\to\; A^{\rm q}_\mu + A^{\rm cl}_\mu$$

where the "classical" fields are fast ($\alpha < \sigma'$) and "quantum" fields are slow ($\alpha > \sigma'$). It should be emphasized that our "classical" field does not satisfy the equation $D^\mu F_{\mu\nu} = 0$; rather, $(D^\mu F^{\rm cl}_{\mu\nu})^a = -g\bar\psi\gamma_\nu t^a\psi$, where $\psi$ are the "classical" (i.e. fast) quark fields. In addition, in this section it is assumed that the slow fields are hard and the fast fields are soft so one can use the light-cone expansion. We will perform calculations in the background-Feynman gauge, where the gluon propagator is $\left(\frac{1}{P^2+2iF}\right)_{\mu\nu}$, see appendix A.



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The first-order term in the expansion of the operator $[\infty, y_*]^{nm}_y F^m_{\bullet i}(y_*, y_\perp)$ in quantum fields has the form

$$[\infty, y_*]^{nm}_y F^m_{\bullet i}(y_*, y_\perp) \stackrel{\rm 1st}{=} \frac{s}{2}\frac{\partial}{\partial y_*}[\infty, y_*]^{nm}_y A^{m\mathrm{q}}_i(y_*, y_\perp) \tag{3.3}$$
$$-[\infty, y_*]^{nm}_y \partial_i A^{m\mathrm{q}}_\bullet(y_*, y_\perp) + i\int_{y_*}^\infty d\frac{2}{s}z'_*\ [\infty, z'_*]_y A^{\mathrm{q}}_\bullet(z'_*, y_\perp)[z'_*, y_*]^{nm}_y F^m_{\bullet i}(y_*, y_\perp)$$

(to save space, we omit the label ${}^{\rm cl}$ from classical fields). The corresponding vertex of gluon emission is given by

$$\lim_{k^2\to 0} k^2\langle A^{a\mathrm{q}}_\mu(k)\big([\infty, y_*]^{nm}_y F^m_{\bullet i}(y_*, y_\perp)\big)^{\rm 1st}\rangle \tag{3.4}$$
$$= \lim_{k^2\to 0} k^2\Big[\frac{s}{2}\frac{\partial}{\partial y_*}[\infty, y_*]^{nm}_y\langle A^{a\mathrm{q}}_\mu(k) A^{m\mathrm{q}}_i(y_*, y_\perp)\rangle - [\infty, y_*]^{nm}_y\langle A^{a\mathrm{q}}_\mu(k)\partial_i A^{m\mathrm{q}}_\bullet(y_*, y_\perp)\rangle$$
$$+ i\frac{2}{s}\int_{y_*}^\infty dz_*\ [\infty, z_*]_y\langle A^{a\mathrm{q}}_\mu(k) A^{\mathrm{q}}_\bullet(z_*, y_\perp)\rangle[z_*, y_*]^{nm}_y F^m_{\bullet i}(y_*, y_\perp)\Big]$$

To calculate the r.h.s. we can use formulas (A.47)–(A.48) from appendix A. As we mentioned, we need contributions to production part of the kernel with the collinear twist up to two. However, it is easy to see that the light-cone expansion of gluon emission vertex starts with the operators of twist one ($\sim F_{\bullet i}$) since the gauge links in the first term in eq. (A.20) cancel in eq. (3.4) and the remaining background-free emission of gluon is proportional to $\delta(\beta_B + \frac{k_\perp^2}{\alpha s})$ which vanishes for $\beta_B > 0$. Thus, to get the contribution to the production part of the kernel of collinear twist up to two it is sufficient to use formula (A.20) for Feynman amplitude and formula (A.23) for complex conjugate amplitude with twist-one (one $F_{\bullet i}$) accuracy. In this case the quark terms do not contribute and the gluon terms simplify to

$$\lim_{k^2\to 0} k^2\langle A^{a\mathrm{q}}_\mu(k) A^{b\mathrm{q}}_\nu(y)\rangle = -ie^{i\frac{k_\perp^2}{\alpha s}y_* - i(k,y)_\perp}\mathcal{O}^{ab}_{\mu\nu}(\infty, y_*, y_\perp; k), \tag{3.5}$$
$$\mathcal{O}_{\mu\nu}(\infty, y_*, y_\perp; k) = g_{\mu\nu}[\infty, y_*]_y + g\int_{y_*}^\infty dz_*\Big(-\frac{4i}{\alpha s^2}k^j(z-y)_* g_{\mu\nu}[\infty, z_*]_y F_{\bullet j}(z_*, y_\perp)[z_*, y_*]_y$$
$$+\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu} - \delta^j_\nu p_{2\mu})[\infty, z_*]_y F_{\bullet j}(z_*, y_\perp)[z_*, y_*]_y\Big)$$

With the help of this formula eq. (3.4) reduces to

$$\lim_{k^2\to 0} k^2\langle A^{a\mathrm{q}}_\mu(k)\Big([\infty, y_*]^{nm}_y F^m_{\bullet i}(y_*, y_\perp)\Big)^{\rm 1st}\rangle \tag{3.6}$$
$$= \Big\{\Big(\frac{k_\perp^2}{2\alpha}g^\perp_{\mu i} - p_{1\mu}k_i\Big)\Big(1 - \frac{4igk^j}{\alpha s^2}\int_{y_*}^\infty dz_*\ (z-y)_*\ [\infty, z_*]_y F_{\bullet j}(z_*, y_\perp)[z_*, \infty]_y\Big)^{an}$$
$$+ig\Big(\frac{2\alpha}{k_\perp^2}p_{1\mu} - \frac{2}{\alpha s}p_{2\mu}\Big)\Big([\infty, y_*]_y F_{\bullet i}(y_*, y_\perp)[y_*, \infty]_y$$
$$-\frac{ik_\perp^2}{\alpha s}\int_{y_*}^\infty dz_*[\infty, z_*]_y F_{\bullet i}(z_*, y_\perp)[z_*, \infty]_y\Big)^{an}$$
$$+\frac{2g}{\alpha s}(g_{\mu i}k^j - \delta^j_\mu k_i)\int_{y_*}^\infty dz_*\ ([\infty, z_*]_y F_{\bullet j}(z_*, y_\perp)[z_*, \infty]_y)^{an}\Big\}e^{i\frac{k_\perp^2}{\alpha s}y_* - i(k,y)_\perp}$$

Note that $k_\mu \times \big(\text{r.h.s. of eq. (3.6)}\big)^\mu = 0$ as required by gauge invariance. Integrating the r.h.s. of eq. (3.6) over $y_*$ we obtain

$$
\begin{aligned}
&L^{ab}_{\mu i}(k, y_\perp, \beta_B) = i \lim_{k^2\to 0} k^2 \langle A^{a\text{q}}_\mu(k)(\mathcal{F}^b_i(\beta_B, y_\perp))^{\text{1st}}\rangle = \frac{2ge^{-i(k,y)_\perp}}{\alpha\beta_B s + k_\perp^2} \qquad (3.7)\\
&\times\left[\frac{\alpha\beta_B s}{k_\perp^2}\left(\frac{k_\perp^2}{\alpha s}p_{2\mu} - \alpha p_{1\mu}\right)\delta^l_i - \delta^l_\mu k_i + \frac{\alpha\beta_B s g_{\mu i} k^l}{k_\perp^2 + \alpha\beta_B s} + \frac{2\alpha k_i k^l}{k_\perp^2 + \alpha\beta_B s}p_{1\mu}\right]\mathcal{F}^{ab}_l\left(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\right)
\end{aligned}
$$

At this point it is convenient to switch to the light-like gauge $p_2^\mu A_\mu = 0$. Since $k_\mu \times \big(\text{r.h.s. of eq. (3.7)}\big)^\mu = 0$ it is sufficient to replace $\alpha p_1^\mu$ in the r.h.s. of eq. (3.7) by $\alpha p_1^\mu - k^\mu = -k_\perp^\mu - \frac{k_\perp^2}{\alpha s}p_2^\mu$. One obtains

$$
\begin{aligned}
&L^{ab}_{\mu i}(k, y_\perp, \beta_B)^{\text{light-like}} = 2ge^{-i(k,y)_\perp} \qquad (3.8)\\
&\times\left[\frac{k^\perp_\mu \delta^l_i}{k_\perp^2} - \frac{\delta^l_\mu k_i + \delta^l_i k^\perp_\mu - g_{\mu i}k^l}{\alpha\beta_B s + k_\perp^2} - \frac{k_\perp^2 g_{\mu i}k^l + 2k^\perp_\mu k_i k^l}{(\alpha\beta_B s + k_\perp^2)^2}\right]\mathcal{F}^{ab}_l\left(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\right) + O(p_{2\mu})
\end{aligned}
$$

We do not write down the terms $\sim p_{2\mu}$ since they do not contribute to the production kernel ($\sim$ square of the expression in the r.h.s. of eq. (3.8)).

For the complex conjugate amplitude one obtains from eq. (A.49)

$$
\begin{aligned}
&-i \lim_{k^2\to 0} k^2 \langle \tilde{A}^a_\mu(x)\tilde{A}^b_\nu(k)\rangle = e^{-i\frac{k_\perp^2}{\alpha s}x_* + i(k,x)_\perp}\tilde{\mathcal{O}}^{ab}_{\mu\nu}(x_*, \infty, x_\perp; k),\\
&\tilde{\mathcal{O}}_{\mu\nu}(x_*, \infty, x_\perp; k) = g_{\mu\nu}[x_*, \infty]_x + g\int_{x_*}^\infty dz_*\, [x_*, z_*]_x\bigg(\frac{4i}{\alpha s^2}(z-x)_* g_{\mu\nu}\tilde{F}_{\bullet j}(z_*, x_\perp)[z_*, \infty]_x k^j\\
&\qquad\qquad - \frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu} - \delta^j_\nu p_{2\mu})\tilde{F}_{\bullet j}(z_*, x_\perp)[z_*, \infty]_x\bigg) \qquad (3.9)
\end{aligned}
$$

where $\tilde{\mathcal{O}}_{\mu\nu}$ is obtained from the eq. (A.23) with twist-two accuracy (as we mentioned, quark operators start from twist two and therefore do not contribute to the production kernel).

Repeating steps which lead us to eq. (3.8) we obtain

$$
\begin{aligned}
&\tilde{L}^{ab}_{i\mu}(k, x_\perp, \beta_B)^{\text{light-like}} = -i \lim_{k^2\to 0} k^2 \langle\big(\tilde{\mathcal{F}}^a_i(\beta_B, x_\perp)\big)^{\text{1st}}\tilde{A}^{b\text{q}}_\mu(k)\rangle^{\text{light-like}} = 2ge^{i(k,x)_\perp} \qquad (3.10)\\
&\times\left[\frac{k^\perp_\mu \delta^k_i}{k_\perp^2} - \frac{\delta^k_\mu k_i + k^\perp_\mu \delta^k_i - g_{\mu i}k^k}{\alpha\beta_B s + k_\perp^2} - \frac{k_\perp^2 g_{\mu i}k^k + 2k^\perp_\mu k_i k^k}{(\alpha\beta_B s + k_\perp^2)^2}\right]\tilde{\mathcal{F}}^{ab}_k\left(\beta_B + \frac{k_\perp^2}{\alpha s}, x_\perp\right) + O(p_{2\mu})
\end{aligned}
$$

The product of Lipatov vertices (3.8) and (3.10) integrated according to eq. (3.1) gives the production part of the evolution kernel in the light-cone limit. To get the full kernel, we need to add the virtual contribution coming from diagrams of figure 1b type.

### 3.2 Virtual part of the kernel

To get the virtual part coming from diagrams of figure 1b type we need to expand the operator $\mathcal{F}$ up to the second order in quantum field

$$\begin{aligned}
&\langle[\infty,y_*]_y^{nm}F^m_{\bullet i}(y_*,y_\perp)\rangle^{\rm 2nd} \qquad (3.11)\\
&=-\frac{4g^2}{s^2}\int_{y_*}^\infty dz_*\int_{y_*}^{z_*}dz'_*\ \big([\infty,z_*]_y\langle A^{\rm q}_\bullet(z_*,y_\perp)[z_*,z'_*]_yA^{\rm q}_\bullet(z'_*,y_\perp)[z'_*,y_*]_y\rangle\big)^{nm}F^m_{\bullet i}(y_*,y_\perp)\\
&\quad+ig\frac{\partial}{\partial y_*}\int_{y_*}^\infty dz_*\big([\infty,z_*]_y\langle A^{\rm q}_\bullet(z_*,y_\perp)[z_*,y_*]_y\big)^{nm}A^{m{\rm q}}_i(y_*,y_\perp)\rangle\\
&\quad-\frac{2ig}{s}\int_{y_*}^\infty dz_*\big([\infty,z_*]_y\langle A^{\rm q}_\bullet(z_*,y_\perp)[z_*,y_*]_y\big)^{nm}\partial_iA^{m{\rm q}}_\bullet(y_*,y_\perp)\rangle
\end{aligned}$$

As we mentioned above, we are interested in operators up to (collinear) twist one. Looking at the explicit expressions for propagators in appendix A it is easy to see that the only contribution of twist one comes from $\langle A^{\rm q}_\bullet(z_*,y_\perp)A^{\rm q}_i(y_*,y_\perp)\rangle$ propagator, which is given by eq. (A.45) with

$$\mathcal{G}_{\bullet i}(z_*,y_*;p_\perp)\ =-\frac{2g}{\alpha s}\int_{y_*}^{z_*}dz'_*\ [z_*,z'_*]F_{\bullet i}(z'_*)[z'_*,y_*] \qquad (3.12)$$

We obtain

$$\begin{aligned}
\langle[\infty,y_*]_y^{nm}F^m_{\bullet i}(y_*,y_\perp)\rangle^{\rm 2nd}=g^2N_c\int_0^\infty\frac{{\text{đ}}\alpha}{\alpha}\Big[&-(y_\perp|\frac{1}{p_\perp^2}|y_\perp)[\infty,y_*]^{nm}F^m_{\bullet i}(y_*,y_\perp) \qquad (3.13)\\
&+\frac{i}{\alpha s}\int_{y_*}^\infty dz_*\ (y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}|y_\perp)[\infty,z_*]^{nm}F^m_{\bullet i}(z_*,y_\perp)\Big]
\end{aligned}$$

where we used Schwinger's notations

$$(x_\perp|f(p_\perp)|y_\perp)\equiv\int{\text{đ}}^2p_\perp\ e^{i(p,x-y)_\perp}f(p),\qquad (x_\perp|p_\perp)=e^{i(p,x)_\perp} \qquad (3.14)$$

For the operator $\mathcal{F}(\beta_B,y_\perp)$ the eq. (3.13) gives

$$\langle\mathcal{F}^n_i(\beta_B,y_\perp)\rangle^{\rm 2nd}=-g^2N_c\int_0^\infty\frac{{\text{đ}}\alpha}{\alpha}(y_\perp|\frac{\alpha\beta_Bs}{p_\perp^2(\alpha\beta_Bs+p_\perp^2)}|y_\perp)\mathcal{F}^n_i(\beta_B,y_\perp) \qquad (3.15)$$

For the complex conjugate amplitude

$$\begin{aligned}
&\langle\tilde F^m_{\bullet i}(x_*,x_\perp)[x_*,\infty]_x^{mn}\rangle^{\rm 2nd} \qquad (3.16)\\
&=-\frac{4g^2}{s^2}\int_{x_*}^\infty dz_*\int_{z_*}^\infty dz'_*\ \tilde F^m_{\bullet i}(x_*,x_\perp)\big([x_*,z_*]_x\langle\tilde A^{\rm q}_\bullet(z_*,x_\perp)[z_*,z'_*]_x\tilde A^{\rm q}_\bullet(z'_*,x_\perp)\rangle[z'_*,\infty]_x\big)^{mn}\\
&\quad-ig\frac{\partial}{\partial x_*}\int_{x_*}^\infty dz_*\langle\tilde A^{m{\rm q}}_i(x_*,x_\perp)\big([x_*,z_*]_x\tilde A^{\rm q}_\bullet(z_*,x_\perp)[z_*,\infty]_x\big)^{mn}\rangle\\
&\quad+\frac{2i}{s}g\int_{x_*}^\infty dz_*\langle\partial_i\tilde A^{m{\rm q}}_\bullet\big([x_*,z_*]_x\tilde A^{\rm q}_\bullet(z_*,x_\perp)[z_*,\infty]_x\big)^{mn}\rangle
\end{aligned}$$

Again, the only contribution of twist one comes from $\langle \tilde{A}_i^{\mathrm{q}}(x_*, x_\perp)\tilde{A}_\bullet^{\mathrm{q}}(z_*, x_\perp)\rangle$ given by eq. (A.46) with

$$\tilde{\mathcal{G}}_{i\bullet}^{kl}(x_*, z_*; p_\perp) = -\frac{2g}{\alpha s}\int_{x_*}^{z_*} dz'_* \ \left([x_*, z'_*]\tilde{F}_{\bullet i}(z'_*)[z'_*, z_*]\right)^{kl} \tag{3.17}$$

(see eq. (A.23)) so the virtual correction in the complex conjugate amplitude is proportional to

$$\langle \tilde{\mathcal{F}}_i^n(\beta_B, x_\perp)\rangle^{2\mathrm{nd}} = -g^2 N_c \int_0^\infty \frac{\text{đ}\alpha}{\alpha}(x_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s + p_\perp^2)}|x_\perp)\tilde{\mathcal{F}}_i^n(\beta_B, x_\perp) \tag{3.18}$$

The total virtual correction is

$$\begin{aligned}
&\langle \tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)\rangle_{\mathrm{virt}} \\
&= -2g^2 N_c \tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)\int_0^\infty \frac{\text{đ}\alpha}{\alpha}\int \text{đ}^2 p_\perp \frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s + p_\perp^2)}
\end{aligned} \tag{3.19}$$

Note that with our rapidity cutoff in $\alpha$ (eq. (2.1)) the contribution (3.19) coming from the diagram in figure 1b is UV finite. Indeed, regularizing the IR divergence with a small gluon mass $m^2$ we obtain

$$\int_0^\sigma \frac{d\alpha}{\alpha}\int d^2 p_\perp \frac{\alpha\beta_B s}{(p_\perp^2 + m^2)(\alpha\beta_B s + p_\perp^2 + m^2)} \simeq \frac{\pi}{2}\ln^2\frac{\sigma\beta_B s + m^2}{m^2} \tag{3.20}$$

which is finite without any UV regulator (the IR divergence is canceled with the corresponding term in the real correction, see eq. (3.24) below). This feature — simultaneous regularization of UV and rapidity divergence — is a consequence of our specific choice of cutoff in rapidity. For a different rapidity cutoff we may have the UV divergence in the remaining integrals which has to be regulated with suitable UV cutoff (for example, see refs. [29, 30]). Let us illustrate this using the example of the figure 1b diagram calculated above. Technically, we calculated the loop integral in this diagram

$$\int \text{đ}\alpha \text{đ}\beta \text{đ}\beta' \text{đ}^2 p_\perp \frac{-\beta_B s}{(\beta - i\epsilon)(\beta' + \beta_B - i\epsilon)(\alpha\beta s - p_\perp^2 - m^2 + i\epsilon)(\alpha\beta' s - p_\perp^2 - m^2 + i\epsilon)} \tag{3.21}$$

by taking residues in the integrals over Sudakov variables $\beta$ and $\beta'$ and cutting the obtained integral over $\alpha$ from above by the cutoff (2.1). Instead, let us take the residue over $\alpha$:

$$\begin{aligned}
&i\beta_B \int \frac{\text{đ}^2 p_\perp}{m^2 + p_\perp^2}\int \text{đ}\beta \text{đ}\beta' \frac{\theta(\beta)\theta(-\beta') - \theta(-\beta)\theta(\beta')}{(\beta' + \beta_B - i\epsilon)(\beta - i\epsilon)(\beta' - \beta)} \\
&= i\beta_B \int \frac{\text{đ}^2 p_\perp}{m^2 + p_\perp^2}\int \frac{\text{đ}\beta \text{đ}\beta'}{\beta' + \beta_B - i\epsilon}\left[\frac{\theta(\beta)\theta(-\beta') - \theta(-\beta)\theta(\beta')}{(\beta - i\epsilon)(\beta' - \beta)} + \frac{\theta(\beta')}{(\beta - i\epsilon)(\beta' - \beta + i\epsilon)}\right] \\
&= i\beta_B \int \frac{\text{đ}^2 p_\perp}{m^2 + p_\perp^2}\int \frac{\text{đ}\beta \text{đ}\beta'}{\beta' + \beta_B - i\epsilon}\frac{\theta(\beta)}{(\beta - i\epsilon)(\beta' - \beta + i\epsilon)} = \beta_B \int \frac{\text{đ}^2 p_\perp}{m^2 + p_\perp^2}\int_0^\infty \frac{\text{đ}\beta}{\beta(\beta + \beta_B)}
\end{aligned} \tag{3.22}$$

which is integral (3.20) with the replacement of variable $\beta = \frac{p_\perp^2}{\alpha s}$.

A conventional way of rewriting this integral in the framework of collinear factorization approach is

$$\beta_B\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int_0^\infty\frac{d\beta}{\beta(\beta+\beta_B)}=\int\frac{d\!\!{}^-{}^2p_\perp}{m^2+p_\perp^2}\int_0^1\frac{dz}{1-z} \tag{3.23}$$

where $z=\frac{\beta_B}{\beta_B+\beta}$ is a fraction of momentum $(\beta_B+\beta)p_2$ of "incoming gluon" (described by $\mathcal{F}_i$ in our formalism) carried by the emitted "particle" with fraction $\beta_Bp_2$, see the discussion of the DGLAP kernel in the next section. Now, if we cut the rapidity of the emitted gluon by cutoff in fraction of momentum $z$, we would still have the UV divergent expression which must be regulated by a suitable UV cutoff.

### 3.3 Evolution kernel in the light-cone limit

Summing the product of Lipatov vertices (3.8) and (3.10) (integrated according to eq. (3.1)) and the virtual correction (3.19) we obtain the one-loop evolution kernel in the light-cone approximation

$$\begin{aligned}&\big(\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)\big)^{\ln\sigma} \qquad (3.24)\\&=2g^2N_c\int_{\sigma'}^{\sigma}\frac{d\!\!{}^-\alpha}{\alpha}d\!\!{}^-{}^2k_\perp\,\bigg\{e^{i(k,x-y)_\perp}\tilde{\mathcal{F}}_k^a\Big(\beta_B+\frac{k_\perp^2}{\alpha s},x_\perp\Big)\mathcal{F}_l^a\Big(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp\Big)\\&\quad\times\bigg[\frac{\delta_i^k\delta_j^l}{k_\perp^2}-\frac{2\delta_i^k\delta_j^l}{\alpha\beta_Bs+k_\perp^2}+\frac{k_\perp^2\delta_i^k\delta_j^l+\delta_j^kk_ik^l+\delta_i^lk_jk^k-\delta_j^lk_ik^k-\delta_i^kk_jk^l-g^{kl}k_ik_j-g_{ij}k^kk^l}{(\alpha\beta_Bs+k_\perp^2)^2}\\&\quad+k_\perp^2\frac{2g_{ij}k^kk^l+\delta_i^kk_jk^l+\delta_j^lk_ik^k-\delta_j^kk_ik^l-\delta_i^lk_jk^k}{(\alpha\beta_Bs+k_\perp^2)^3}-\frac{k_\perp^4g_{ij}k^kk^l}{(\alpha\beta_Bs+k_\perp^2)^4}\bigg]\\&\quad-\frac{\alpha\beta_Bs}{k_\perp^2(\alpha\beta_Bs+k_\perp^2)}\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)\bigg\}^{\ln\sigma'}\end{aligned}$$

where rapidities of gluons in the operators in the r.h.s. are restricted from above by $\ln\sigma'$.

Let us write down now the evolution equation for gluon TMDs defined by the matrix element (2.9). If we define $\beta_B$ as a fraction of the momentum $p$ of the original hadron we have $\beta_B<1$. Moreover, in the production part of the amplitude we have a kinematical restriction that the sum of $\beta_B$ and the fraction carried by emitted gluon $\frac{k_\perp^2}{\alpha s}$ should be less than one. This leads to the upper cutoff in the $k_\perp$ integral $k_\perp^2\leq\alpha(1-\beta_B)s$ and we get the equation

$$\begin{aligned}&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle \qquad (3.25)\\&=\frac{g^2N_c}{\pi}\int d\!\!{}^-{}^2k_\perp\bigg\{e^{i(k,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_k^a\Big(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_l^a\Big(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp\Big)|p\rangle\\&\quad\times\bigg[\frac{\delta_i^k\delta_j^l}{k_\perp^2}-\frac{2\delta_i^k\delta_j^l}{\sigma\beta_Bs+k_\perp^2}+\frac{k_\perp^2\delta_i^k\delta_j^l+\delta_j^kk_ik^l+\delta_i^lk_jk^k-\delta_j^lk_ik^k-\delta_i^kk_jk^l-g^{kl}k_ik_j-g_{ij}k^kk^l}{(\sigma\beta_Bs+k_\perp^2)^2}\\&\quad+k_\perp^2\frac{2g_{ij}k^kk^l+\delta_i^kk_jk^l+\delta_j^lk_ik^k-\delta_j^kk_ik^l-\delta_i^lk_jk^k}{(\sigma\beta_Bs+k_\perp^2)^3}-\frac{k_\perp^4g_{ij}k^kk^l}{(\sigma\beta_Bs+k_\perp^2)^4}\bigg]\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\\&\quad-\frac{\sigma\beta_Bs}{k_\perp^2(\sigma\beta_Bs+k_\perp^2)}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle\bigg\}\end{aligned}$$

(there is obviously no restriction on $k_\perp$ in the virtual diagram).



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If the target hadron is unpolarized one can use the parametrization (2.9)

$$\begin{aligned}
&\langle p|\tilde{\mathcal{F}}_i^a(\beta_B, z_\perp)\mathcal{F}_j^a(\beta_B, 0_\perp)|p+\xi p_2\rangle^\eta \\
&= 2\pi^2\delta(\xi)\beta_B g^2\Big[-g_{ij}\mathcal{D}(\beta_B, z_\perp, \eta) - \frac{1}{m^2}(2\partial_i\partial_j + g_{ij}\partial_\perp^2)\mathcal{H}(\beta_B, z_\perp, \eta)\Big] \\
&= 2\pi^2\delta(\xi)\beta_B g^2\Big[-g_{ij}\mathcal{D}(\beta_B, z_\perp, \eta) - \frac{4}{m^2}(2z_iz_j + g_{ij}z_\perp^2)\mathcal{H}''(\beta_B, z_\perp, \eta)\Big]
\end{aligned} \tag{3.26}$$

where $\eta \equiv \ln\sigma$, $\mathcal{H}(\beta_B, z_\perp, \eta) \equiv \int d\!\!\!{}^-{}^2k_\perp\ e^{-i(k,z)_\perp}\mathcal{H}(\beta_B, k_\perp, \eta)$ and $\mathcal{H}''(\beta_B, z_\perp, \eta) \equiv \left(\frac{\partial}{\partial z^2}\right)^2\mathcal{H}(\beta_B, z_\perp, \eta)$. Rewriting eq. (3.25) in terms of variable $\beta \equiv \frac{k_\perp^2}{\sigma s}$ one obtains

$$\begin{aligned}
&\frac{d}{d\eta}\Big[g_{ij}\alpha_s\mathcal{D}(\beta_B, z_\perp, \eta) + \frac{4}{m^2}(2z_iz_j + g_{ij}z_\perp^2)\alpha_s\mathcal{H}''(\beta_B, z_\perp, \eta)\Big] \qquad (3.27)\\
&= \frac{\alpha_s N_c}{\pi}\int_0^{1-\beta_B} d\beta\ \Big\{g_{ij}J_0\big(|z_\perp|\sqrt{\sigma\beta s}\big)\alpha_s\mathcal{D}(\beta_B+\beta, z_\perp, \eta) \\
&\quad\times\Big[\frac{\beta_B+\beta}{\beta\beta_B} - \frac{2}{\beta_B} + \frac{3\beta}{\beta_B(\beta_B+\beta)} - \frac{2\beta^2}{\beta_B(\beta_B+\beta)^2} + \frac{\beta^3}{\beta_B(\beta_B+\beta)^3}\Big] \\
&\quad+ J_2(|z_\perp|\sqrt{\sigma\beta s})\Big(2\frac{z_iz_j}{z_\perp^2} + g_{ij}\Big)\alpha_s\mathcal{D}(\beta_B+\beta, z_\perp, \eta)\frac{\beta}{\beta_B(\beta_B+\beta)} \\
&\quad+ \frac{4}{m^2}J_0\big(|z_\perp|\sqrt{\sigma\beta s}\big)(2z_iz_j + g_{ij}z_\perp^2)\alpha_s\mathcal{H}''(\beta_B+\beta, z_\perp, \eta)\Big[\frac{\beta_B+\beta}{\beta_B\beta} - \frac{2}{\beta_B} + \frac{\beta}{\beta_B(\beta_B+\beta)}\Big] \\
&\quad+ \frac{4g_{ij}}{m^2}z_\perp^2 J_2\big(|z_\perp|\sqrt{\sigma\beta s}\big)\alpha_s\mathcal{H}''(\beta_B+\beta, z_\perp, \eta) \\
&\quad\times\Big[\frac{\beta}{\beta_B(\beta_B+\beta)} - \frac{2\beta^2}{\beta_B(\beta_B+\beta)^2} + \frac{\beta^3}{\beta_B(\beta_B+\beta)^3}\Big]\Big\} \\
&\quad- \frac{\alpha_s N_c}{\pi}\int_0^\infty d\beta\frac{\beta_B}{\beta(\beta_B+\beta)}\Big[g_{ij}\alpha_s\mathcal{D}(\beta_B, z_\perp, \eta) + \frac{4}{m^2}(2z_iz_j + g_{ij}z_\perp^2)\alpha_s\mathcal{H}''(\beta_B, z_\perp, \eta)\Big]
\end{aligned}$$

where we used the formula

$$\frac{1}{2\pi}\int d\theta e^{i(k,z)_\perp}[2k_ik_j + k_\perp^2 g_{ij}] = -J_2(kz)k_\perp^2\left(\frac{2}{z_\perp^2}z_iz_j + g_{ij}\right) \tag{3.28}$$

The evolution equation (3.27) can be rewritten as a system of evolution equations for $\mathcal{D}$



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and $\mathcal{H}''$ functions ($z' \equiv \frac{\beta_B}{\beta+\beta_B}$):

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(\beta_B,z_\perp,\eta) \tag{3.29}$$
$$= \frac{\alpha_s N_c}{\pi}\int_{\beta_B}^1 \frac{dz'}{z'}\Big\{J_0\Big(|z_\perp|\sqrt{\sigma s\beta_B\frac{1-z'}{z'}}\Big)\Big[\Big(\frac{1}{1-z'}\Big)_+ + \frac{1}{z'} - 2 + z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{\beta_B}{z'},z_\perp,\eta\Big)$$
$$+\frac{4}{m^2}(1-z')z'z_\perp^2 J_2\Big(|z_\perp|\sqrt{\sigma s\beta_B\frac{1-z'}{z'}}\Big)\alpha_s\mathcal{H}''\Big(\frac{\beta_B}{z'},z_\perp,\eta\Big)\Big\},$$
$$\frac{d}{d\eta}\alpha_s\mathcal{H}''(\beta_B,z_\perp,\eta)$$
$$= \frac{\alpha_s N_c}{\pi}\int_{\beta_B}^1 \frac{dz'}{z'}\Big\{J_0\Big(|z_\perp|\sqrt{\sigma s\beta_B\frac{1-z'}{z'}}\Big)\Big[\Big(\frac{1}{1-z'}\Big)_+ - 1\Big]\alpha_s\mathcal{H}''\Big(\frac{\beta_B}{z'},z_\perp,\eta\Big)$$
$$+\frac{m^2}{4z_\perp^2}\frac{1-z'}{z'}J_2\Big(|z_\perp|\sqrt{\sigma s\beta_B\frac{1-z'}{z'}}\Big)\alpha_s\mathcal{D}\Big(\frac{\beta_B}{z'},z_\perp,\eta\Big)\Big\}$$

where $\int_x^1 dz f(z)g(z)_+ = \int_x^1 dz f(z)g(z) - \int_0^1 dz f(1)g(z)$. The above equation is our final result for the rapidity evolution of gluon TMDs in the near-light-cone case.

It is instructive to check that the evolution equation (3.29) agrees with the (one-loop) DGLAP kernel. If we take the light-cone limit $x_\perp = y_\perp$ ($\Leftrightarrow z_\perp = 0$) we get

$$\frac{d}{d\eta}\alpha_s\mathcal{D}(\beta_B,0_\perp,\eta) = \frac{\alpha_s}{\pi}N_c\int_{\beta_B}^1 \frac{dz'}{z'}\Big[\Big(\frac{1}{1-z'}\Big)_+ + \frac{1}{z'} - 2 + z'(1-z')\Big]\alpha_s\mathcal{D}\Big(\frac{\beta_B}{z'},0_\perp,\eta\Big) \tag{3.30}$$

One immediately recognizes the expression in the square brackets as gluon-gluon DGLAP kernel (the term $\frac{11}{12}\delta(1-z')$ is absent since we consider the gluon light-ray operator multiplied by an extra $\alpha_s$). It should be mentioned, however, that eq. (3.30) is not a proper DGLAP equation since the latter is formulated for the gluon parton density on the light cone defined by

$$d_g(x_B,\ln\mu^2) = -\frac{1}{8\pi^2\alpha_s(p\cdot n)x_B}\int du\; e^{-ix_B u(pn)}\langle p|\tilde{\mathcal{F}}_i^a(un)\mathcal{F}^{ai}(0)|p\rangle^\mu \tag{3.31}$$

where the light-ray gluon operator $F_i^a(un)[un,0]F^{ai}(0)$ is regularized with counterterms at normalization point $\mu^2$ (recall that on the light ray T-product of operators coincide with the usual product).

Comparing eqs. (2.4) and (3.31) we see that $d_g(x_B) = \mathcal{D}(x_B,z_\perp=0)$ modulo different cutoffs: by counterterms for $d_g(x_B,\mu^2)$ and by "brute force" rapidity cutoff $Y<\eta$ in $D(x_B,z_\perp=0,\eta=\ln\sigma)$. However, with the leading-log accuracy subtracting the counterterms is equivalent to imposing a cutoff in transverse momenta of the emitted gluons $k_\perp^2<\mu^2$. If we would calculate the leading-order renorm-group equation for the light-ray operator $\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,x_\perp)$ we would cut the integral over $k_\perp^2$ from above by $\mu^2$ and leave the integration over rapidity ($\alpha$) unrestricted. Thus, we would obtain

$$\begin{aligned}
&\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^\mu \qquad (3.32)\\
&=\frac{\alpha_s}{\pi}N_c\int_0^\infty\frac{d\alpha}{\alpha}\int_{\frac{\mu'^2}{\alpha s}}^{\frac{\mu^2}{\alpha s}}d\beta\,\Big\{\theta(1-\beta_B-\beta)\Big[\frac{1}{\beta}-\frac{2\beta_B}{(\beta_B+\beta)^2}+\frac{\beta_B^2}{(\beta_B+\beta)^3}-\frac{\beta_B^3}{(\beta_B+\beta)^4}\Big]\\
&\quad\times\langle p|\tilde{\mathcal{F}}_i^n(\beta_B+\beta,x_\perp)\mathcal{F}^{ni}(\beta_B+\beta,x_\perp)|p\rangle^{\mu'}-\frac{\beta_B}{\beta(\beta_B+\beta)}\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^{\mu'}\Big\}\\
&=\frac{\alpha_s}{\pi}N_c\int_0^\infty d\beta\int_{\frac{\mu'^2}{\beta s}}^{\frac{\mu^2}{\beta s}}\frac{d\alpha}{\alpha}\,\Big\{\theta(1-\beta_B-\beta)\Big[\frac{1}{\beta}-\frac{2\beta_B}{(\beta_B+\beta)^2}+\frac{\beta_B^2}{(\beta_B+\beta)^3}-\frac{\beta_B^3}{(\beta_B+\beta)^4}\Big]\\
&\quad\times\langle p|\tilde{\mathcal{F}}_i^n(\beta_B+\beta,x_\perp)\mathcal{F}^{ni}(\beta_B+\beta,x_\perp)|p\rangle^{\mu'}-\frac{\beta_B}{\beta(\beta_B+\beta)}\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^{\mu'}\Big\}
\end{aligned}$$

which should be compared to eq. (3.24) with $x_\perp=y_\perp$

$$\begin{aligned}
&\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^{\ln\sigma}\\
&=\frac{\alpha_s}{\pi}N_c\int_{\sigma'}^{\sigma}\frac{d\alpha}{\alpha}\int_0^\infty d\beta\,\Big\{\theta(1-\beta_B-\beta)\Big[\frac{1}{\beta}-\frac{2\beta_B}{(\beta_B+\beta)^2}+\frac{\beta_B^2}{(\beta_B+\beta)^3}-\frac{\beta_B^3}{(\beta_B+\beta)^4}\Big]\\
&\quad\times\langle p|\tilde{\mathcal{F}}_i^n(\beta_B+\beta,x_\perp)\mathcal{F}^{ni}(\beta_B+\beta,x_\perp)|p\rangle^{\ln\sigma'}-\frac{\beta_B\beta^{-1}}{\beta_B+\beta}\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^{\ln\sigma'}\Big\}
\end{aligned}$$

In the leading log approximation $\beta\sim\beta_B=x_B$ so one can replace the cutoff $\frac{\mu^2}{\beta s}$ in eq. (3.32) by the cutoff $\frac{\mu^2}{x_B s}=\sigma$ and hence $d_g(x_B,\ln\mu^2)=D(x_B,z_\perp=0,\ln\sigma s)$ with the leading-log accuracy. The equation (3.32) can be rewritten as an evolution equation

$$\begin{aligned}
&\mu^2\frac{d}{d\mu^2}\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^\mu \qquad (3.33)\\
&=\frac{\alpha_s(\mu)}{\pi}N_c\int_0^\infty d\beta\,\Big\{\theta(1-\beta_B-\beta)\Big[\frac{1}{\beta}-\frac{2\beta_B}{(\beta_B+\beta)^2}+\frac{\beta_B^2}{(\beta_B+\beta)^3}-\frac{\beta_B^3}{(\beta_B+\beta)^4}\Big]\\
&\quad\times\langle p|\tilde{\mathcal{F}}_i^n(\beta_B+\beta,x_\perp)\mathcal{F}^{ni}(\beta_B+\beta,x_\perp)|p\rangle^\mu-\frac{\beta_B}{\beta(\beta_B+\beta)}\langle p|\tilde{\mathcal{F}}_i^n(\beta_B,x_\perp)\mathcal{F}^{ni}(\beta_B,x_\perp)|p\rangle^\mu\Big\}
\end{aligned}$$

which can be transformed to the standard DGLAP form [31–33]

$$\begin{aligned}
&\mu^2\frac{d}{d\mu^2}\alpha_s(\mu)d_g(x_B,\ln\mu^2) \qquad (3.34)\\
&=\frac{\alpha_s(\mu)}{\pi}N_c\int_{\beta_B}^1\frac{dz'}{z'}\,\Big[\Big(\frac{1}{1-z'}\Big)_+ +\frac{1}{z'}-2+z'(1-z')\Big]\alpha_s(\mu)d_g\Big(\frac{\beta_B}{z'},\ln\mu^2\Big)
\end{aligned}$$

There is a subtle point in comparison of our rapidity evolution of light-ray operators to the conventional $\mu^2$ evolution described by renorm-group equations: the self-energy diagrams are not regulated by our rapidity cutoff so the $\delta$-function terms in our version of the DGLAP equations are absent.[3] Indeed, in our analysis we do not change the UV treatment of the theory, we just define the Wilson-line (or light-ray) operators by the requirement that

[3]For eq. (3.34) the absence of these terms is accidental, due to an extra $\alpha_s$ in the definition (2.4).

gluons emitted by those operators have rapidity cutoff (2.1). The UV divergences in self-energy and other internal loop diagrams appearing in higher-order calculations are absorbed in the usual $Z$-factors. So, in a way, we will have two evolution equations for our operators: the trivial $\mu^2$ evolution described by anomalous dimensions of corresponding gluon (or quark) fields and the rapidity evolution. Combined together, the two should describe the $Q^2$ evolution of DIS structure functions. Presumably, the argument of coupling constant in LO equation (3.30) (which is $\mu^2$ by default) will be replaced by $\sigma\beta_B s$ in accordance with common lore that this argument is determined by characteristic transverse momenta.[4] We plan to return to this point in the future NLO analysis.

## 4 Evolution kernel in the general case

In this section we will find the leading-order rapidity evolution of gluon operator (2.8)

$$\big(\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)\big)^{\ln\sigma}$$

with the rapidity cutoff $Y < \eta = \ln\sigma$ for all emitted gluons. As we mentioned in the Introduction, in order to find the evolution kernel we need to integrate over slow gluons with $\sigma > \alpha > \sigma'$ and temporarily freeze fast fields with $\alpha < \sigma'$ to be integrated over later. To this end we need the one-loop diagrams in the fast background fields with arbitrary transverse momenta. In the previous section we have found the evolution kernel in background fields with transverse momenta $l_\perp \ll p_\perp$ where $p_\perp$ is a characteristic momentum of our quantum slow fields. In this section at first we will find the Lipatov vertex and virtual correction for the case $l_\perp \sim p_\perp$ and then write down general formulas which are correct in the whole region of the transverse momentum.

The key observation is that for transverse momenta of quantum and background field of the same order we can use the shock-wave approximation developed for small-$x$ physics. To find the evolution kernel we consider the operator (2.8) in the background of external field $A_\bullet(x_*, x_\perp)$ (the absence of $x_\bullet$ in the argument corresponds to $\alpha = 0$). Moreover, we assume that the background field $A_\bullet(x_*, x_\perp)$ has a narrow support and vanishes outside the $[-\sigma_*, \sigma_*]$ interval. This is obviously not the most general form of the external field, but it turns out that after obtaining the kernel of the evolution equation it is easy to restore the result for any background field by insertion of gauge links at $\pm\infty p_1$, see the discussion after eq. (5.4).

Since the typical $\beta$'s of the external field are $\beta_{\rm fast} \sim \frac{l_\perp^2}{\alpha_{\rm fast}s}$ the support of the shock wave $\sigma_*$ is of order of $\frac{1}{\beta_{\rm fast}} \sim \frac{\sigma' s}{l_\perp^2}$. This is to be compared to the typical scale of slow fields $\frac{1}{\beta_{\rm slow}} \sim \frac{\alpha s}{p_\perp^2} \gg \sigma_*$ so we see that the fast background field can be approximated by a narrow shock wave. In the "pure" low-x case $\beta_B = 0$ one can assume that the support of this shock wave is infinitely narrow. As we shall see below, in our case of arbitrary $\beta_B$ we need to

[4] Note that while in the usual renorm-group DGLAP the argument of coupling constant is a part of LO equation, with our cutoff this argument can be determined only at the NLO level, same as in the case of NLO BK equation at low $x$ [17–20]. This is not surprising since we use the rapidity cutoff borrowed from the NLO BK analysis.



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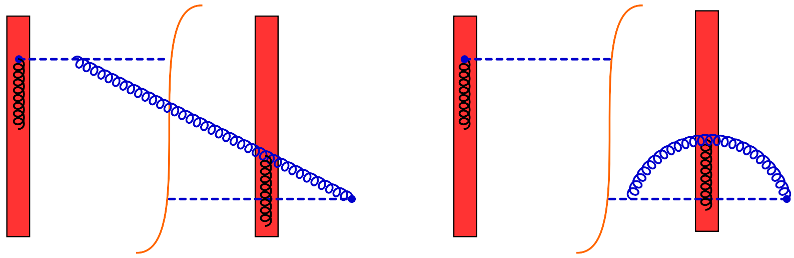


*(a)* *(b)*

**Figure 3**. Typical diagrams for production (a) and virtual (b) contributions to the evolution kernel. The shaded area denotes shock wave of background fast fields.



look inside the shock wave so we will separate all integrals over longitudinal distances $z_*$ in parts "inside the shock wave" $|z_*| < \sigma_*$ and "outside the shock wave" $|z_*| > \sigma_*$, calculate them separately and check that the sum of "inside" and "outside" contributions does not depend on $\sigma_*$ with our accuracy.

### 4.1 Production part of the evolution kernel

In the leading order there is only one extra gluon and we get the typical diagrams of figure 3 type. The production part of the kernel can be obtained as a square of a Lipatov vertex - the amplitude of the emission of a real gluon by the operator $\mathcal{F}_i^a$ (see eq. (3.1))

$$\langle\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)\rangle^{\ln\sigma} = -\int_{\sigma'}^{\sigma}\frac{d\!\!\!^-\alpha}{2\alpha}d\!\!\!^-{}^2k_\perp\,\big(\tilde{L}_i^{ba;\mu}(k, x_\perp, \beta_B)L_{\mu j}^{ab}(k, y_\perp, \beta_B)\big)^{\ln\sigma'} \tag{4.1}$$

where the Lipatov vertices of gluon emission are defined as

$$\begin{aligned} L_{\mu i}^{ab}(k, y_\perp, \beta_B) &= i\lim_{k^2\to 0}k^2\langle A_\mu^a(k)\mathcal{F}_i^b(\beta_B, y_\perp)\rangle \\ \tilde{L}_{i\mu}^{ba}(k, x_\perp, \beta_B) &= -i\lim_{k^2\to 0}k^2\langle\tilde{\mathcal{F}}_i^b(\beta_B, x_\perp)\tilde{A}_\mu^a(k)\rangle \end{aligned} \tag{4.2}$$

(cf. eqs. (3.2) and (3.10)). Hereafter $\langle\mathcal{O}\rangle$ means the average of operator $\mathcal{O}$ in the shock-wave background.

### 4.2 Lipatov vertex of gluon emission in the shock wave background

As we discussed above, we calculate the diagrams in the background of a shock wave of width $\sim\frac{\sigma' s}{l_\perp^2}$ where $l_\perp$ is the characteristic transverse momentum of the external shock-wave field. Note that the factor in the exponent in the definition of $\mathcal{F}(\beta_B)$ is $\sim\beta_B\frac{\sigma' s}{l_\perp^2}$ which is not necessarily small at various $\beta_B$ and $l_\perp^2$ and therefore we need to take into account the diagram in figure 4c with emission point inside the shock wave. We will do this in a following way: we assume that all of the shock wave is contained within $\sigma_* > z_* > -\sigma_*$, calculate diagrams in figure 4a–d and check that the dependence on $\sigma_*$ cancels in the final result for the sum of these diagrams.

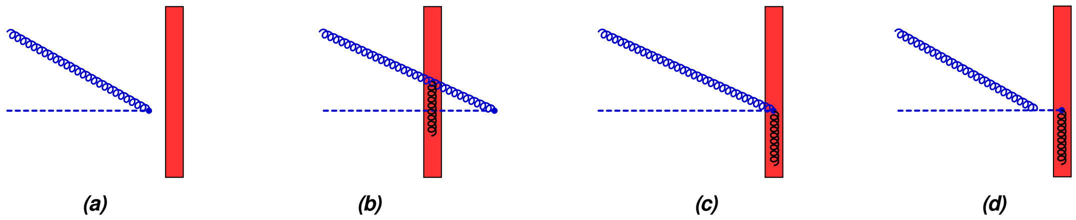



**Figure 4**. Lipatov vertex of gluon emission.

We start the calculation with the expansion of the gluon fields in $\mathcal{F}(\beta_B, z_\perp)$ in the first order in slow "quantum" field:

$$\mathcal{F}_i^n(\beta_B, y_\perp) \stackrel{1st}{=} g\int d\frac{2}{s}y_*\ e^{i\beta_B y_*}\Big[-i\frac{s}{2}\beta_B[\infty, y_*]_y^{nm}A_i^{\mathrm{q}m}(y_*, y_\perp)$$
$$-[\infty, y_*]_y^{nm}\partial_i A_\bullet^{\mathrm{q}m}(y_*, y_\perp)+\frac{2ig}{s}\int_{y_*}^\infty dz_*[\infty, z_*]_y A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, y_*]_y^{nm}F_{\bullet i}^m(y_*, y_\perp)\Big] \tag{4.3}$$

where the gauge links and $F_{\bullet i}^m$ are made of fast "external" fields. The corresponding vertex of gluon emission is given by

$$\lim_{k^2\to 0} k^2\langle A_\mu^{a\mathrm{q}}(k)\mathcal{F}_i^n(\beta_B, y_\perp)^{1\mathrm{st}}\rangle = \lim_{k^2\to 0} k^2 g\int dy_*\ e^{i\beta_B y_*}\langle A_\mu^{a\mathrm{q}}(k)\Big\{-i\beta_B[\infty, y_*]_y^{nm}A_i^{m\mathrm{q}}(y_*, y_\perp)$$
$$-\frac{2}{s}[\infty, y_*]_y^{nm}\partial_i A_\bullet^{m\mathrm{q}}(y_*, y_\perp)+\frac{4ig}{s^2}\int_{y_*}^\infty dz_*\big([\infty, z_*]A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, y_*]\big)^{nm}F_{\bullet i}^m(y_*, y_\perp)\Big\}\rangle \tag{4.4}$$

The diagrams in figure 4a, 4b, and 4(c-d) correspond to different regions of integration over $y_*$ in eq. (4.3): $y_* > \sigma_*$, $-\sigma_* > y_*$, and $\sigma_* > y_* > -\sigma_*$, respectively.

The trivial calculation of figure 4a contribution yields

$$i\lim_{k^2\to 0} k^2\langle A_\mu^{aq}(k)\mathcal{F}_i^n(\beta_B, y_\perp)\rangle_{\text{figure 4a}} \tag{4.5}$$
$$= ig\int_{\sigma_*}^\infty dy_*\ e^{i\beta_B y_*}[\infty, y_*]^{nm}\lim_{k^2\to 0} k^2\langle A_\mu^a(k)\Big[-i\beta_B A_i^m(y_*, y_\perp)-\frac{2}{s}\partial_i A_\bullet^m(y_*, y_\perp)\Big]\rangle$$
$$= g\delta^{an}\frac{g_{\mu i}\alpha\beta_B s+2\alpha k_i p_{1\mu}}{\alpha\beta_B s+k_\perp^2}\ e^{i(\beta_B+\frac{k_\perp^2}{\alpha s})\sigma_*-i(k,y)_\perp}$$

### 4.2.1 Diagram in figure 4b

Next step is the calculation of figure 4b contribution. Using the vertex of gluon emission from the shock wave (B.30) one obtains

$$ig\lim_{k^2\to 0}\int_{-\infty}^{-\sigma_*} dy_*\ e^{i\beta_B y_*}k^2\langle A_\mu^{a\mathrm{q}}(k)[\infty, y_*]_y^{nm}\Big\{-i\beta_B A_i^{m\mathrm{q}}(y_*, y_\perp)-\frac{2}{s}\partial_i A_\bullet^{m\mathrm{q}}(y_*, y_\perp)\Big\}\rangle$$
$$= g\int_{-\infty}^{-\sigma_*} dy_*\ e^{i\beta_B y_*}\int d^2z_\perp\ e^{-i(k,z)_\perp}\big\{-i\beta_B\mathcal{O}_{\mu i}(\infty, y_*, z_\perp; k)(z_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)$$
$$-\frac{2i}{s}\mathcal{O}_{\mu\bullet}(\infty, y_*, z_\perp; k)(z_\perp|p_i e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)\big\}^{am}(U_y^\dagger)^{mn} \tag{4.6}$$

where $\mathcal{O}$ is given by eqs. (B.31):

$$
\begin{aligned}
&\mathcal{O}^{ab}_{\mu\nu}(\infty,y_*,z_\perp;k) \stackrel{y_*<-\sigma_*}{=} g_{\mu\nu}U^{ab}_z\\
&+g\int_{-\infty}^{\infty}dz_*\Big([\infty,z_*]_z\Big[-\frac{2iz_*}{\alpha s^2}g_{\mu\nu}(2k^j-iD^j)+\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\Big]F_{\bullet j}(z_*,z_\perp)[z_*,-\infty]_z\Big)^{ab}\\
&+\frac{4g}{\alpha^2s^3}\int_{-\infty}^{\infty}dz_*\Big([\infty,z_*]_z\Big\{-ip_{2\mu}p_{2\nu}D^jF_{\bullet j}(z_*,z_\perp)[z_*,-\infty]_z\\
&+g\int_{-\infty}^{z_*}dz'_*\ \Big[2i\alpha g_{\mu\nu}z'_*-\frac{4}{s}p_{2\mu}p_{2\nu}\Big]\ F_{\bullet j}(z_*,z_\perp)[z_*,z'_*]_zF_\bullet{}^j(z'_*,z_\perp)[z'_*,-\infty]_z\Big\}\Big)^{ab}\\
&+\frac{g^2}{\alpha s^2}\Big\{\int_{-\infty}^{\infty}dz_*\int_{-\infty}^{z_*}dz'_*\bar\psi(z_*,z_\perp)[z_*,\infty]_zt^aU_zt^b[-\infty,z'_*]_z\gamma^\perp_\mu\ \not{p}_1\gamma^\perp_\nu\psi(z'_*,z_\perp)\\
&-\int_{-\infty}^{\infty}dz_*\int_{z_*}^{\infty}dz'_*\bar\psi(z_*,z_\perp)\gamma^\perp_\nu\ \not{p}_1\gamma^\perp_\mu[z_*,-\infty]_zt^bU^\dagger_zt^a[\infty,z'_*]_z\psi(z'_*,z_\perp)\Big\}
\end{aligned}
\tag{4.7}
$$

where we replaced $y_*$ by $-\infty$ since we assumed that there is no gauge field outside the $[-\sigma_*,\sigma_*]$ interval.

Let us compare relative size of terms in the r.h.s. of this equation. The leading $g_{\mu\nu}$ term is $\sim U_z\sim 1$ and it is clear that all other $g_{\mu\nu}$ terms are small. Indeed, the first term in the second line is $\sim \frac{g}{\alpha s^2}\int dz_*\ z_*(2k^j-iD^j)F_{\bullet j}\sim\frac{\sigma_*}{\alpha s}k^j\partial_jU\sim\frac{k_\perp^2}{\alpha s}\sigma_*\ll 1$ since the width of the shock wave is $\sim\frac{s\sigma'}{k_\perp^2}$ and $\alpha\gg\sigma'$ (recall that in this section $l_\perp\sim k_\perp$). Similarly, the first term in the fourth line is $\sim\frac{g^2}{\alpha s^3}\int dz_*dz'_*\ z'_*F_{\bullet j}(z_*)F_\bullet{}^j(z'_*)\sim\frac{\sigma_*}{\alpha s}\partial^jU\partial_jU\sim\sigma_*\frac{k_\perp^2}{\alpha s}\ll 1$.

Next, let us find out the relative size of quark terms in eq. (4.7). The "power counting" for external quark fields in comparison to gluon ones is $\frac{g^2}{s}\int dz_*\bar\psi\ \not{p}_1\psi(z_*)\sim\frac{g}{s}\int dz_*D^iF_{\bullet i}(z_*)\sim k_\perp^2U\sim k_\perp^2$ and each extra integration inside the shock wave brings extra $\sigma_*$. Thus, the two last terms in eq. (4.7) are $\sim g^\perp_{\mu\nu}\frac{\sigma_*}{\alpha s}k_\perp^2\ll 1$.[5]

After omitting small terms the expression (4.7) reduces to

$$
\begin{aligned}
\mathcal{O}_{\mu\nu}(\infty,-\infty,z_\perp;k)&=g_{\mu\nu}U_z+\frac{4g}{\alpha s^2}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\int_{-\infty}^{\infty}dz_*\ [\infty,z_*]_zF_{\bullet j}(z_*,z_\perp)[z_*,-\infty]_z\\
&\quad+\frac{4g}{\alpha^2s^3}p_{2\mu}p_{2\nu}\int_{-\infty}^{\infty}dz_*\ [\infty,z_*]_z\Big\{-iD^jF_{\bullet j}(z_*,z_\perp)[z_*,-\infty]_z\\
&\quad-\frac{4g}{s}\int_{-\infty}^{z_*}dz'_*\ F_{\bullet j}(z_*,z_\perp)[z_*,z'_*]_zF_\bullet{}^j(z'_*,z_\perp)[z'_*,-\infty]_z\Big\}\\
&=g_{\mu\nu}U_z+\frac{2i}{\alpha s}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\partial_jU_z-\frac{2p_{2\mu}p_{2\nu}}{\alpha^2s^2}\partial_\perp^2U_z
\end{aligned}
\tag{4.8}
$$

where we used the formula

$$
\begin{aligned}
\partial_\perp^2U_z=&g\int_{-\infty}^{\infty}dz_*\ [\infty,z_*]_z\\
&\times\Big(\frac{2i}{s}D^jF_{\bullet j}(z_*,z_\perp)[z_*,-\infty]_z+\frac{8g}{s^2}\int_{-\infty}^{z_*}dz'_*\ F_{\bullet j}(z_*,z_\perp)[z_*,z'_*]_zF_\bullet{}^j(z'_*,z_\perp)[z'_*,-\infty]_z\Big)
\end{aligned}
\tag{4.9}
$$

[5]Note, however, that the quark term $\sim\frac{g}{\alpha^2s^3}p_{2\mu}p_{2\nu}\int dz_*D^jF_{\bullet j}(z_*)\sim p_{2\mu}p_{2\nu}\frac{k_\perp^2}{\alpha^2s^2}U$ is of the same order of magnitude as the gluon term $\sim\frac{g^2}{\alpha^2s^4}p_{2\mu}p_{2\nu}\int dz_*dz'_*F_{\bullet j}(z_*)F_\bullet{}^j(z'_*)$.

Using eq. (4.8) one obtains for the r.h.s. of eq. (4.6)

$$
\begin{aligned}
& i\lim_{k^2\to 0} k^2\langle A^{aq}_\mu(k)\mathcal{F}^n_i(\beta_B,y_\perp)\rangle_{\text{figure 4b}} \qquad (4.10)\\
&= g\int d^2z_\perp\, e^{-i(k,z)_\perp}\Big\{-\mathcal{O}_{\mu i}(\infty,-\infty,z_\perp;k)(z_\perp|\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}e^{-i(\beta_B+\frac{p_\perp^2}{\alpha s})\sigma_*}|y_\perp)\\
&\quad -\frac{2}{s}\mathcal{O}_{\mu\bullet}(\infty,-\infty,z_\perp;k)(z_\perp|\frac{p_i\alpha s}{\alpha\beta_Bs+p_\perp^2}e^{-i(\beta_B+\frac{p_\perp^2}{\alpha s})\sigma_*}|y_\perp)\Big\}^{am}(U^\dagger_y)^{mn}\\
&= -ge^{-i\beta_B\sigma_*}(k_\perp|\Big[g_{\mu i}U-\frac{2ip_{2\mu}}{\alpha s}\partial_iU\\
&\quad +\frac{2}{\beta_Bs}(p_{1\mu}U+\frac{i}{\alpha}\partial^\perp_\mu U-\frac{p_{2\mu}}{\alpha^2s}\partial_\perp^2U)p_i\Big]\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}U^\dagger|y_\perp)^{an}
\end{aligned}
$$

where we used the fact that $\frac{p_\perp^2}{\alpha s}\sigma_*\ll 1$ when all the transverse momenta are of the same order.

### 4.2.2 Diagrams in figure 4c,d

Next step is the calculation of figure 4 c,d contributions. Using the vertex of gluon emission from the shock wave (A.47) and eqs. (B.6), (B.7) one obtains

$$
\begin{aligned}
& \lim_{k^2\to 0} k^2ig\int_{-\sigma_*}^{\sigma_*}dy_*\, e^{i\beta_By_*}\langle A^{aq}_\mu(k)\{-i\beta_B[\infty,y_*]^{nm}_yA^{mq}_i(y_*,y_\perp) \qquad (4.11)\\
& -\frac{2}{s}[\infty,y_*]^{nm}_y\partial_iA^{mq}_\bullet(y_*,y_\perp)+\frac{4ig}{s^2}\int_{y_*}^\infty dz_*\,\big([\infty,z_*]_yA^q_\bullet(z_*,y_\perp)[z_*,y_*]_y\big)^{nm}F^m_{\bullet i}(y_*,y_\perp)\}\rangle\\
&= g\int_{-\sigma_*}^{\sigma_*}dy_*\,\Big\{e^{i(\beta_B+\frac{k_\perp^2}{\alpha s})y_*}\Big[-i\beta_B\mathcal{O}^{am}_{\mu i}(\infty,y_*,y_\perp;k)-\frac{2}{s}\Big(ik_i+\frac{\partial}{\partial y^i}\Big)\mathcal{O}^{am}_{\mu\bullet}(\infty,y_*,y_\perp;k)\Big]\\
&\quad +\frac{4ig}{s^2}\int_{y_*}^\infty dz_*\, e^{i\beta_By_*+i\frac{k_\perp^2}{\alpha s}z_*}\big(\mathcal{O}_{\mu\bullet}(\infty,z_*,y_\perp;k)[z_*,y_*]_yF_{\bullet i}(y_*,y_\perp)\big)^{am}\Big\}[y_*,\infty]^{mn}_ye^{-i(k,y)_\perp}\\
&= ge^{-i(k,y)_\perp}\int_{-\sigma_*}^{\sigma_*}dy_*\Big[e^{i\beta_By_*}\Big(-i\Big\{\beta_B\mathcal{O}_{\mu i}(\infty,y_*,y_\perp;k)+\frac{2}{s}k_i\mathcal{O}_{\mu\bullet}(\infty,y_*,y_\perp;k)\Big\}[y_*,\infty]_y\\
&\quad -\frac{2}{s}\frac{\partial}{\partial y^i}\big(\mathcal{O}_{\mu\bullet}(\infty,y_*,y_\perp;k)[y_*,\infty]_y\big)+\frac{4ig}{s^2}\int_{y_*}^\infty dz_*\,\mathcal{O}_{\mu\bullet}(\infty,y_*,y_\perp;k)[y_*,z_*]_yF_{\bullet i}(z_*,y_\perp)\\
&\quad \times[z_*,\infty]_y\Big)+\frac{4ig}{s^2}\int_{y_*}^\infty dz_*\, e^{i\beta_By_*+i\frac{k_\perp^2}{\alpha s}z_*}\mathcal{O}_{\mu\bullet}(\infty,z_*,y_\perp;k)[z_*,y_*]_yF_{\bullet i}(y_*,y_\perp)[y_*,\infty]_y\Big]^{an}
\end{aligned}
$$

where $\mathcal{O}_{\mu\nu}=\mathcal{G}_{\mu\nu}+\mathcal{Q}_{\mu\nu}+\bar{\mathcal{Q}}_{\mu\nu}$ and $\mathcal{G}$, $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ are given by eqs. (B.6) and (B.7). As we mentioned above, the contributions with extra $(z-\sigma)_*$ are small and so are the quark terms (except term $\sim D^jF_{\bullet j}$). So, we have $\mathcal{Q}_{\mu i}=\bar{\mathcal{Q}}_{\mu i}=0$ and

$$
\begin{aligned}
& \mathcal{O}_{\mu i}(\infty,y_*,y_\perp;k)[y_*,\infty]_y=g_{\mu i}-\frac{4gp_{2\mu}}{\alpha s^2}\int_{y_*}^\infty dz_*\,[\infty,z_*]_yF_{\bullet i}(z_*,y_\perp)[z_*,\infty]_y,\\
& \mathcal{O}_{\mu\bullet}(\infty,y_*,y_\perp;k)[y_*,\infty]_y=p_{1\mu}+g\int_{y_*}^\infty dz_*[\infty,z_*]_y\Big\{\frac{2}{\alpha s}F_{\bullet\mu}(z_*,y_\perp)[z_*,\infty]_y \qquad (4.12)\\
& -\frac{2ip_{2\mu}}{\alpha^2s^2}D^jF_{\bullet j}(z_*,y_\perp)[z_*,\infty]_y-\frac{8gp_{2\mu}}{\alpha^2s^3}\int_{y_*}^{z_*}dz'_*F_{\bullet j}(z_*,y_\perp)[z_*,z'_*]_yF_\bullet{}^j(z'_*,y_\perp)[z'_*,\infty]_y\Big\}
\end{aligned}
$$

After some algebra the r.h.s. of eq. (4.11) reduces to

$$
\begin{aligned}
& i \lim_{k^2\to 0} k^2\langle A^{aq}_\mu(k)\mathcal{F}^n_i(\beta_B,y_\perp)\rangle_{\text{figure 4c+d}} \\
&= -\frac{g_{\mu i}\alpha\beta_B s+2\alpha k_i p_{1\mu}}{\alpha\beta_B s+k_\perp^2}\Big[e^{i\left(\beta_B+\frac{k_\perp^2}{\alpha s}\right)\sigma_*}-e^{-i\left(\beta_B+\frac{k_\perp^2}{\alpha s}\right)\sigma_*}\Big]e^{-i(k,y)_\perp}g\delta^{an} \\
&\quad +ge^{-i(k,y)_\perp}\bigg\{\frac{2}{\alpha\beta_B s}\big(\alpha p_{1\mu}\delta^j_i+\beta_B p_{2\mu}\delta^j_i-k_i\delta^j_\mu\big)\big[\mathcal{F}_j(\beta_B,y_\perp)-i\partial_j U_y U^\dagger_y e^{-i\beta_B\sigma_*}\big] \\
&\quad -\frac{2\alpha p_{1\mu}}{k_\perp^2}\mathcal{F}_i(\beta_B,y_\perp)+\frac{2p_{2\mu}k_i}{\alpha^2 s^2\beta_B}\big[\mathcal{V}(\beta_B,y_\perp)-e^{-i\beta_B\sigma_*}\partial_\perp^2 U_y U^\dagger_y\big] \\
&\quad +\frac{2i}{\alpha\beta_B s}\partial^y_i\Big\{\mathcal{F}_\mu(y_\perp,\beta_B)-i\partial_\mu U_y U^\dagger_y e^{-i\beta_B\sigma_*}-\frac{p_{2\mu}}{\alpha s}\big[\mathcal{V}(\beta_B,y_\perp)-e^{-i\beta_B\sigma_*}\partial_\perp^2 U_y U^\dagger_y\big]\Big\} \\
&\quad +\frac{4g}{\beta_B s^2}\int dz_* dz'_*\; e^{i\beta_B \min(z_*,z'_*)} \\
&\quad \times[\infty,z_*]\Big\{\frac{2}{\alpha s}F_{\bullet\mu}(z_*,y_\perp)[z_*,z'_*]_y-\frac{2ip_{2\mu}}{\alpha^2 s^2}D^jF_{\bullet j}(z_*,y_\perp)[z_*,z'_*]_y\Big\}F_{\bullet i}(z'_*,y_\perp)[z'_*,\infty]_y \\
&\quad -\frac{32g^2p_{2\mu}}{\alpha^2 s^5\beta_B}\int dz_* dz'_* dz''_*\;\theta(z_*-z''_*)e^{i\beta_B\min(z'_*,z''_*)}[\infty,z_*]_y F_{\bullet j}(z_*,y_\perp)[z_*,z''_*]_y F_\bullet{}^j(z''_*,y_\perp) \\
&\quad \times[z''_*,z'_*]_y F_{\bullet i}(z'_*,y_\perp)[z'_*,\infty]_y-\frac{2}{\alpha\beta_B s}e^{-i\beta_B\sigma_*}\Big(\partial_\mu U_y\partial_i U^\dagger_y+\frac{ip_{2\mu}}{\alpha s}\partial_\perp^2 U_y\partial_i U^\dagger_y\Big)\bigg\}^{an}
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
\mathcal{V}(\beta_B,y_\perp) \equiv g\int_{-\infty}^{\infty} dz_*\; e^{i\beta_B z_*}\bigg(&\frac{2i}{s}[\infty,z_*]_y D^jF_{\bullet j}(z_*,y_\perp) \\
&+\frac{8g}{s^2}\int_{z_*}^{\infty} dz'_*\;[\infty,z'_*]_y F_{\bullet j}(z'_*,y_\perp)[z'_*,z_*]_y F_\bullet{}^j(z_*,y_\perp)\bigg)[z_*,\infty]_y
\end{aligned}
\tag{4.14}
$$

### 4.2.3 Lipatov vertex

The sum of eqs. (4.5), (4.10), and (4.13) gives the Lipatov vertex of gluon emission in the form

$$
\begin{aligned}
& L^{ab}_{\mu i}(k,y_\perp,\beta_B) = g\delta^{ab}\frac{g_{\mu i}\alpha\beta_B s+2\alpha k_i p_{1\mu}}{\alpha\beta_B s+k_\perp^2}\;e^{-i\beta_B\sigma_*-i(k,y)_\perp} \\
& -ge^{-i\beta_B\sigma_*}(k_\perp|\Big[g_{\mu i}U-\frac{2ip_{2\mu}}{\alpha s}\partial_i U+\big(p_{1\mu}U+\frac{i}{\alpha}\partial^\perp_\mu U-\frac{p_{2\mu}}{\alpha^2 s}\partial_\perp^2 U\big)\frac{2p_i}{\beta_B s}\Big]\frac{\alpha\beta_B s}{\alpha\beta_B s+p_\perp^2}U^\dagger|y_\perp)^{ab} \\
& +ge^{-i(k,y)_\perp}\bigg\{\frac{2}{\alpha\beta_B s}\big(\alpha p_{1\mu}\delta^j_i+\beta_B p_{2\mu}\delta^j_i-k_i\delta^j_\mu\big)\big[\mathcal{F}_j(\beta_B,y_\perp)-i\partial_j U_y U^\dagger_y e^{-i\beta_B\sigma_*}\big] \\
& -\frac{2\alpha p_{1\mu}}{k_\perp^2}\mathcal{F}_i(\beta_B,y_\perp)+\frac{2p_{2\mu}k_i}{\alpha^2 s^2\beta_B}\big[\mathcal{V}(\beta_B,y_\perp)-\partial_\perp^2 U_y U^\dagger_y e^{-i\beta_B\sigma_*}\big] \\
& +\frac{2i}{\alpha\beta_B s}\partial^y_i\Big\{\mathcal{F}_\mu(\beta_B,y_\perp)-i\partial_\mu U_y U^\dagger_y e^{-i\beta_B\sigma_*}-\frac{p_{2\mu}}{\alpha s}\big[\mathcal{V}(\beta_B,y_\perp)-e^{-i\beta_B\sigma_*}\partial_\perp^2 U_y U^\dagger_y\big]\Big\} \\
& +\frac{4g}{\beta_B s^2}\int dz_* dz'_*\; e^{i\beta_B\min(z_*,z'_*)} \\
& \times[\infty,z_*]\Big\{\frac{2}{\alpha s}F_{\bullet\mu}(z_*,y_\perp)[z_*,z'_*]_y-\frac{2ip_{2\mu}}{\alpha^2 s^2}D^jF_{\bullet j}(z_*,y_\perp)[z_*,z'_*]_y\Big\}F_{\bullet i}(z'_*,y_\perp)[z'_*,\infty]_y
\end{aligned}
$$

$$-\frac{32g^2p_{2\mu}}{\alpha^2 s^5\beta_B}\int dz_*dz'_*dz''_*\ \theta(z_*-z''_*)e^{i\beta_B\min(z'_*,z''_*)}[\infty,z_*]_yF_{\bullet j}(z_*,y_\perp)[z_*,z''_*]_yF_\bullet{}^j(z''_*,y_\perp)$$
$$\times[z''_*,z'_*]_yF_{\bullet i}(z'_*,y_\perp)[z'_*,\infty]_y-\frac{2}{\alpha\beta_B s}e^{-i\beta_B\sigma_*}\Big(\partial_\mu U_y\partial_i U_y^\dagger+\frac{ip_{2\mu}}{\alpha s}\partial_\perp^2U_y\partial_iU_y^\dagger\Big)\Big\}^{ab} \tag{4.15}$$

This expression explicitly depends on the cutoff $\sigma_*$. However, we can set $\sigma_*=0$ in the r.h.s. of eq. (4.15) (and eliminate few terms as well). To demonstrate this, let us consider two cases: $\beta_B\ll\frac{1}{\sigma_*}$ and $\beta_B\geq\frac{1}{\sigma_*}$. In the first case

$$\begin{aligned}L^{ab}_{\mu i}(k,y_\perp,\beta_B)\ &\stackrel{\beta_B\sigma_*\ll 1}{=}\ -ge^{-i(k,y)_\perp}\frac{2\alpha p_{1\mu}}{k_\perp^2}\mathcal{F}^{ab}_i(\beta_B,y_\perp)+g(k_\perp|\frac{g_{\mu i}\alpha\beta_Bs+2\alpha k_ip_{1\mu}}{\alpha\beta_Bs+k_\perp^2} \\ &-\Big[g_{\mu i}U-\frac{2ip_{2\mu}}{\alpha s}\partial_iU+\frac{2}{\beta_Bs}\Big(p_{1\mu}U+\frac{i}{\alpha}\partial^\perp_\mu U-\frac{p_{2\mu}}{\alpha^2s}\partial_\perp^2U\Big)p_i\Big]\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}U^\dagger|y_\perp)^{ab} \\ &=g(k_\perp|g_{\mu i}\Big(\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}-U\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}U^\dagger\Big)+2\alpha p_{1\mu}\Big(\frac{p_i}{\alpha\beta_Bs+p_\perp^2}-U\frac{p_i}{\alpha\beta_Bs+p_\perp^2}U^\dagger\Big) \\ &+\Big[\frac{2ip_{2\mu}}{\alpha s}\partial_iU-\frac{2i}{\alpha\beta_Bs}\partial_\mu^\perp Up_i+\frac{2p_{2\mu}}{\alpha^2s^2\beta_B}\partial_\perp^2Up_i\Big]\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}U^\dagger-\frac{2\alpha p_{1\mu}}{p_\perp^2}i(\partial_iU)U^\dagger|y_\perp)^{ab}\end{aligned} \tag{4.16}$$

and all other terms are small since they contain extra factors $e^{i\beta_Bz_*}-e^{-i\beta_B\sigma_*}\simeq(z-\sigma)_*$ (or $z'_*-\sigma_*$ or $z''_*-\sigma_*$) in the integrand.

In the second case $\sigma'\beta_Bs\geq p_\perp^2$ so $\alpha\beta_Bs\gg p_\perp^2$ and we get

$$\begin{aligned}&L^{ab}_{\mu i}(k,y_\perp,\beta_B)\\ &=e^{-i(k,y)_\perp}\Big(2\Big(\frac{p_{2\mu}}{\alpha s}-\frac{\alpha p_{1\mu}}{k_\perp^2}\Big)\mathcal{F}_i(\beta_B,y_\perp)-\frac{2k_i}{\alpha\beta_Bs}\mathcal{F}_\mu(\beta_B,y_\perp)+\frac{2i}{\alpha\beta_Bs}\partial_i^y\mathcal{F}_\mu(\beta_B,y_\perp)\\ &\quad+\frac{8g^2}{\alpha\beta_Bs^3}\int dz_*dz'_*\ e^{i\beta_B\min(z_*,z'_*)}\ [\infty,z_*]F_{\bullet\mu}(z_*,y_\perp)[z_*,z'_*]_yF_{\bullet i}(z'_*,y_\perp)[z'_*,\infty]_y\Big)^{ab}\end{aligned} \tag{4.17}$$

where we used the formula

$$\begin{aligned}&(k_\perp|g_{\mu i}\Big(\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}-U\frac{\alpha\beta_Bs}{\alpha\beta_Bs+p_\perp^2}U^\dagger\Big)+2\alpha p_{1\mu}\Big(\frac{p_i}{\alpha\beta_Bs+p_\perp^2}-U\frac{p_i}{\alpha\beta_Bs+p_\perp^2}U^\dagger\Big)|y_\perp)\\ &=(k_\perp|\frac{\alpha\beta_Bsg_{\mu i}+2\alpha p_{1\mu}k_i}{\alpha\beta_Bs+k_\perp^2}(2ik^j\partial_jU-\partial_\perp^2U)\frac{1}{\alpha\beta_Bs+p_\perp^2}U^\dagger+2i\alpha p_{1\mu}\partial_iU\frac{1}{\alpha\beta_Bs+p_\perp^2}U^\dagger|y_\perp)\end{aligned} \tag{4.18}$$

Let us now compare the contributions of various terms in the r.h.s. of eq. (4.15) to the production part of the evolution kernel defined by the square of Lipatov vertices (4.15). It is clear that the square of the first term $\sim\big(\frac{p_{2\mu}}{\alpha s}-\frac{\alpha p_{1\mu}}{k_\perp^2}\big)\mathcal{F}_i$ is proportional to $\tilde{\mathcal{F}}^i\frac{1}{k_\perp^2}\mathcal{F}_i$ and contributions of all other terms are down by at least one power of $\frac{p_\perp^2}{\alpha\beta_Bs}$. Thus, with our accuracy

$$L^{ab}_{\mu i}(k,y_\perp,\beta_B)\ \stackrel{\alpha\beta_Bs\gg p_\perp^2}{=}\ 2e^{-i(k,y)_\perp}\Big(\frac{p_{2\mu}}{\alpha s}-\frac{\alpha p_{1\mu}}{k_\perp^2}\Big)\mathcal{F}^{ab}_i(\beta_B,y_\perp) \tag{4.19}$$

We see that in both cases (4.16) and (4.19) one can replace $\sigma_*$ by 0. Moreover, with our accuracy the Lipatov vertex (4.15) can be reduced to the "direct sum" of eqs. (4.16) and (4.19):

$$
\begin{aligned}
L^{ab}_{\mu i}(k,y_\perp,\beta_B) &= 2ge^{-i(k,y)_\perp}\left(\frac{p_{2\mu}}{\alpha s}-\frac{\alpha p_{1\mu}}{k_\perp^2}\right)[\mathcal{F}_i(\beta_B,y_\perp)-U_i(y_\perp)]^{ab} \qquad (4.20)\\
&+g(k_\perp|g_{\mu i}\left(\frac{\alpha\beta_B s}{\alpha\beta_B s+p_\perp^2}-U\frac{\alpha\beta_B s}{\alpha\beta_B s+p_\perp^2}U^\dagger\right)+2\alpha p_{1\mu}\left(\frac{p_i}{\alpha\beta_B s+p_\perp^2}-U\frac{p_i}{\alpha\beta_B s+p_\perp^2}U^\dagger\right)\\
&+\left[2i\beta_B p_{2\mu}\partial_i U-2i\partial^\perp_\mu U p_i+\frac{2p_{2\mu}}{\alpha s}\partial_\perp^2 U p_i\right]\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger-\frac{2\alpha p_{1\mu}}{p_\perp^2}U_i|y_\perp)^{ab}
\end{aligned}
$$

where we introduced the notation $U_i\equiv\mathcal{F}_i(0)=i(\partial_i U)U^\dagger$. It is clear that at $\beta_B\sigma_*\ll 1$ the first term in the r.h.s. of this equation disappears and we get the r.h.s. of eq. (4.16). On the other hand, as we saw above, at $\beta_B\sigma_*\geq 1$ all terms in the last two lines in the r.h.s. of eq. (4.20) are small except $(k_\perp|\frac{2p_{2\mu}}{\alpha s}U_i-\frac{2\alpha p_{1\mu}}{p_\perp^2}U_i|y_\perp)^{ab}$ which cancels the second term in the first line of eq. (4.20) so we get the r.h.s. of eq. (4.19). It is worth noting that at $\beta_B=0$ eq. (4.20) agrees with the Lipatov vertex obtained in ref. [34].

It is instructive to check the Lipatov vertex property $k^\mu L^{ab}_{\mu i}(k,y_\perp,\beta_B)=0$. One obtains

$$
\begin{aligned}
&k^\mu\times(\text{r.h.s. of eq. (4.20)})_\mu \qquad (4.21)\\
&=g(k_\perp|k_i\left(\frac{\alpha\beta_B s}{\alpha\beta_B s+k_\perp^2}-U\frac{\alpha\beta_B s}{\alpha\beta_B s+p_\perp^2}U^\dagger\right)+k_\perp^2\left(\frac{k_i}{\alpha\beta_B s+k_\perp^2}-U\frac{p_i}{\alpha\beta_B s+p_\perp^2}U^\dagger\right)\\
&\quad+(\alpha\beta_B s[p_i,U]+[p_\perp^2+\alpha\beta_B s,U]p_i)\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger-U_i|y_\perp)=0
\end{aligned}
$$

### 4.3 Lipatov vertex for arbitrary transverse momenta

Let us demonstrate that for arbitrary transverse momenta the Lipatov vertex of gluon emission is given by the following "interpolating formula"

$$
\begin{aligned}
&L^{ab}_{\mu i}(k,y_\perp,\beta_B)\\
&=g(k_\perp|\frac{\alpha\beta_B s g_{\mu i}+2\alpha p_{1\mu}k_i}{\alpha\beta_B s+k_\perp^2}(2ik^j\partial_j U-\partial_\perp^2 U)\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger+2i\alpha p_{1\mu}\partial_i U\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger\\
&\quad+\frac{2i}{\alpha s}p_{2\mu}\partial_i U\frac{\alpha\beta_B s}{\alpha\beta_B s+p_\perp^2}U^\dagger-\left[2i\partial_\mu U-\frac{2p_{2\mu}}{\alpha s}\partial_\perp^2 U\right]\frac{p_i}{\alpha\beta_B s+p_\perp^2}U^\dagger-\frac{2\alpha p_{1\mu}}{p_\perp^2}i(\partial_i U)U^\dagger|y_\perp)^{ab}\\
&\quad+\frac{2ge^{-i(k,y)_\perp}}{\alpha\beta_B s+k_\perp^2}\left[-\delta^j_\mu k_i+\frac{2\alpha k_i k^j p_{1\mu}}{\alpha\beta_B s+k_\perp^2}+\frac{\alpha\beta_B s g_{\mu i}k^j}{\alpha\beta_B s+k_\perp^2}+\beta_B p_{2\mu}\delta^j_i-\alpha p_{1\mu}\frac{\alpha\beta_B s}{k_\perp^2}\delta^j_i\right]\\
&\quad\times\left[\mathcal{F}_j\left(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp\right)-U_j(y_\perp)\right]^{ab} \qquad (4.22)
\end{aligned}
$$

Let us consider at first the light-cone limit corresponding to the case when the characteristic transverse momenta of the external "fast" gluon fields are small in comparison to the momenta of "slow" gluons which we integrated over. As we discussed above, the higher-twist terms $\sim D_jF_{\bullet k}$ or $\sim F_{\bullet j}F_{\bullet k}$ exceed our accuracy so we can eliminate terms $\sim \partial_\perp^2 U$ and commute operators $\partial_j U$ with $\frac{1}{p_\perp^2+\alpha\beta_B s}$ resulting in

$$\begin{aligned}
&\text{r.h.s. of eq. (4.22)} \stackrel{\text{light}-\text{cone}}{=} 2g(k_\perp|k^j\frac{\alpha\beta_B sg_{\mu i}+2\alpha p_{1\mu}k_i}{(\alpha\beta_B s+k_\perp^2)^2}U_j+\frac{\alpha p_{1\mu}}{\alpha\beta_B s+k_\perp^2}U_i\\
&\quad+\frac{p_{2\mu}\beta_B}{\alpha\beta_B s+k_\perp^2}U_i-\frac{k_i}{\alpha\beta_B s+k_\perp^2}U_{\mu_\perp}-\frac{\alpha p_{1\mu}}{k_\perp^2}U_i|y_\perp)^{ab}\\
&\quad+\frac{2ge^{-i(k,y)_\perp}}{\alpha\beta_B s+k_\perp^2}\Big[-\delta^j_\mu k_i+\frac{2\alpha k_ik^jp_{1\mu}}{\alpha\beta_B s+k_\perp^2}+\frac{\alpha\beta_B sg_{\mu i}k^j}{\alpha\beta_B s+k_\perp^2}+\beta_B p_{2\mu}\delta^j_i-\alpha p_{1\mu}\frac{\alpha\beta_B s}{k_\perp^2}\delta^j_i\Big]\\
&\quad\times\Big[\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp\Big)-U_j(y_\perp)\Big]^{ab}
\end{aligned}\tag{4.23}$$

It is clear now that the first two lines in the r.h.s. cancel the last term in the square brackets in the last line so we recover the light-cone result (3.8).

Next we consider the case when the transverse momenta of fast and slow fields are comparable so the Lipatov vertex is given by eq. (4.20) above. The difference between the r.h.s.'s of eq. (4.22) and eq. (4.20) is

$$\frac{2ge^{-i(k,y)_\perp}}{\alpha\beta_B s+k_\perp^2}\Big[-\delta^j_\mu k_i+\frac{2\alpha k_ik^jp_{1\mu}}{\alpha\beta_B s+k_\perp^2}+\frac{\alpha\beta_B sg_{\mu i}k^j}{\alpha\beta_B s+k_\perp^2}\Big]\Big[\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp\Big)-U_j(y_\perp)\Big]^{ab}\tag{4.24}$$

where we used eq. (4.18). It is easy to see that the expression (4.24) is small in both $\beta_B\ll\frac{1}{\sigma_*}$ and $\beta_B\geq\frac{1}{\sigma_*}$ cases. Indeed, when $\beta_B\ll\frac{1}{\sigma_*}$ the integral representing $\mathcal{F}_j\big(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp\big)-U_j(y_\perp)$ contains an extra factor $e^{i(\beta_B+\frac{k_\perp^2}{\alpha s})z_*}-1\sim(\beta_B+\frac{k_\perp^2}{\alpha s})\sigma_*\ll 1$ in the integrand and in the $\beta_B\geq\frac{1}{\sigma_*}$ case the eq. (4.24) is $\sim\frac{k_\perp^2}{\alpha\beta_B s}\ll 1$ in comparison to the leading term in this limit (4.19).

As in the light-cone case, for calculation of the evolution kernel it is convenient to go to the light-like gauge $p_2^\mu A_\mu=0$. Since $k_\mu\times\big(\text{r.h.s. of eq. (4.22)}\big)^\mu=0$ (see eq. (4.21)) it is sufficient to replace $\alpha p_1^\mu$ in the r.h.s. of eq. (4.22) by $\alpha p_1^\mu-k^\mu=-k_\perp^\mu-\frac{k_\perp^2}{\alpha s}p_2^\mu$. One obtains

$$\begin{aligned}
&L^{ab}_{\mu i}(k,y_\perp,\beta_B)^{\text{light}-\text{like}}\\
&=g(k_\perp|\frac{\alpha\beta_B sg_{\mu i}-2k^\perp_\mu k_i}{\alpha\beta_B s+k_\perp^2}(2ik^j\partial_jU-\partial_\perp^2U)\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger-2ik^\perp_\mu\partial_iU\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger\\
&\quad-2i\partial_\mu U\frac{p_i}{\alpha\beta_B s+p_\perp^2}U^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}U_i|y_\perp)^{ab}\\
&\quad+2ge^{-i(k,y)_\perp}\Big[\frac{k^\perp_\mu\delta^j_i}{k_\perp^2}-\frac{\delta^j_\mu k_i+\delta^j_ik^\perp_\mu-g_{\mu i}k^j}{\alpha\beta_B s+k_\perp^2}-\frac{k_\perp^2g_{\mu i}k^j+2k^\perp_\mu k_ik^j}{(\alpha\beta_B s+k_\perp^2)^2}\Big]\\
&\quad\times\Big[\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\alpha s},y_\perp\Big)-U_j(y_\perp)\Big]^{ab}+O(p_{2\mu})
\end{aligned}\tag{4.25}$$

As usual, we do not display the term $\sim p_{2\mu}$ since it does not contribute to the evolution kernel. Using $[p_\perp^2, U] = -2ip^j\partial_j U + \partial_\perp^2 U$ one can rewrite this vertex as

$$
\begin{aligned}
& L^{ab}_{\mu i}(k, y_\perp, \beta_B)^{\rm light-like} = i \lim_{k^2\to 0} k^2 \langle A^{aq}_\mu(k)\mathcal{F}^b_i(\beta_B, y_\perp)\rangle^{\rm light-like} \qquad (4.26)\\
& = g(k_\perp|U\frac{p_\perp^2 g_{\mu i} + 2p^\perp_\mu p_i}{\alpha\beta_B s + p_\perp^2}U^\dagger - \frac{k_\perp^2 g_{\mu i} + 2k^\perp_\mu k_i}{\alpha\beta_B s + k_\perp^2}|y_\perp)^{ab} + \frac{2gk^\perp_\mu}{k_\perp^2}e^{-i(k,y)_\perp}\mathcal{F}^{ab}_i\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\Big)\\
& \quad -2ge^{-i(k,y)_\perp}\Big[\frac{\delta^j_\mu k_i + \delta^j_i k^\perp_\mu - g_{\mu i}k^j}{\alpha\beta_B s + k_\perp^2} + \frac{g_{\mu i}k_\perp^2 k^j + 2k^\perp_\mu k_i k^j}{(\alpha\beta_B s + k_\perp^2)^2}\Big]\breve{\mathcal{F}}^{ab}_j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\Big) + O(p_{2\mu})
\end{aligned}
$$

where we introduced the notation

$$
\breve{\mathcal{F}}_j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\Big) \equiv \mathcal{F}_j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\Big) - \mathcal{F}_j(0, y_\perp) \qquad (4.27)
$$

(recall that $\mathcal{F}_j(0, y_\perp) = U_j(y_\perp) \equiv i\partial_j U_y U_y^\dagger$).

It should be emphasized that while we constructed the Lipatov vertex (4.22) as a formula which interpolates between the light-cone result (3.8) for small transverse momenta of background fields and shock-wave result (4.20) for comparable transverse momenta, we have just demonstrated that with our leading-log accuracy our final expression (4.22) is correct in the whole range of the transverse momenta.

It is convenient to rewrite the Lipatov vertex (4.26) in a different form without explicit subtraction (4.27). Starting from eq. (4.25) we get

$$
\begin{aligned}
& L^{ab}_{\mu i}(k, y_\perp, \beta_B)^{\rm light-like} \qquad (4.28)\\
& = g(k_\perp|\mathcal{F}^j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}\Big)\Big\{\frac{\alpha\beta_B s g_{\mu i} - 2k^\perp_\mu k_i}{\alpha\beta_B s + k_\perp^2}(k_j U + U p_j)\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger\\
& \quad -2k^\perp_\mu U\frac{g_{ij}}{\alpha\beta_B s + p_\perp^2}U^\dagger - 2g_{\mu j}U\frac{p_i}{\alpha\beta_B s + p_\perp^2}U^\dagger + \frac{2k^\perp_\mu}{k_\perp^2}g_{ij}\Big\}|y_\perp)^{ab} + O(p_{2\mu})
\end{aligned}
$$

where the operator $\mathcal{F}_i(\beta)$ is defined as usual

$$
(k_\perp|\mathcal{F}_i(\beta)|y_\perp) \equiv \frac{2}{s}\int dy_*\ e^{i\beta y_* - i(k,y)_\perp}\mathcal{F}_i(y_*, y_\perp) \qquad (4.29)
$$

Let us prove that eq. (4.28) coincides with eq. (4.26) with our accuracy. First, as we discussed above, in the light-cone case ($l_\perp^2 \ll p_\perp^2$) we can drop higher-twist terms and commute operators $U$ with $p_i$ and $\frac{1}{p_\perp^2 + \alpha\beta_B s}$ which gives us eq. (3.8). Second, consider the "shock-wave" case $l_\perp^2 \sim p_\perp^2$. When $\beta_B \ll \frac{1}{\sigma_*}$ the integral representing $\mathcal{F}_j(\beta_B + \frac{k_\perp^2}{\alpha s})$ contains an exponential factor $e^{i(\beta_B + \frac{k_\perp^2}{\alpha s})z_*} \sim e^{i(\beta_B + \frac{k_\perp^2}{\alpha s})\sigma_*}$. This factor can be approximated by one, since $\frac{k_\perp^2}{\alpha s}\sigma_* \ll 1$ in the shock-wave case (see the discussion above), so we can replace $\mathcal{F}_j(\beta_B + \frac{k_\perp^2}{\alpha s})$ by $U_j$ and get

$$
\begin{aligned}
& L^{ab}_{\mu i}(k, y_\perp, \beta_B)^{\rm light-like} \qquad (4.30)\\
& \simeq\ g(k_\perp|\Big\{\frac{\alpha\beta_B s g_{\mu i} - 2k^\perp_\mu k_i}{\alpha\beta_B s + k_\perp^2}(k_j i\partial^j U + i\partial^j U p_j)\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger - 2k^\perp_\mu i\partial_i U\frac{1}{\alpha\beta_B s + p_\perp^2}U^\dagger\\
& \quad -2i\partial_\mu U\frac{p_i}{\alpha\beta_B s + p_\perp^2}U^\dagger + \frac{2k^\perp_\mu}{k_\perp^2}U_i\Big\}|y_\perp)^{ab} + O(p_{2\mu})
\end{aligned}
$$

which gives the first two lines in the r.h.s. of eq. (4.25). As it was shown above, the last two lines in the r.h.s. of eq. (4.25) are small at $\beta_B \ll \frac{1}{\sigma_*}$ so eq. (4.28) coincides with eq. (4.25) at $\beta_B \ll \frac{1}{\sigma_*}$ with our accuracy. Finally, in the $\beta_B \geq \frac{1}{\sigma_*}$ case $\alpha\beta_B s \gg p_\perp^2$ and therefore the eq. (4.28) reduces to

$$L_{\mu i}^{ab}(k, y_\perp, \beta_B)^{\text{light-like}} \simeq \frac{2k_\mu^\perp}{k_\perp^2} e^{-i(k,y)_\perp} g\mathcal{F}_i^{ab}\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, y_\perp\Big) + O(p_{2\mu})$$

which is the same as eq. (4.25) in this limit.

Similar calculation for complex-conjugate amplitude gives

$$\begin{aligned}
&\tilde{L}_{i\mu}^{ba}(k, x_\perp, \beta_B)^{\text{light-like}} = -i \lim_{k^2\to 0} k^2 \langle \tilde{\mathcal{F}}_i^b(\beta_B, x_\perp)\tilde{A}_\mu^{aq}(k)\rangle^{\text{light-like}} \qquad (4.31)\\
&= g(x_\perp|\tilde{U}\frac{p_\perp^2 g_{\mu i} + 2p_\mu^\perp p_i}{\alpha\beta_B s + p_\perp^2}\tilde{U}^\dagger - \frac{p_\perp^2 g_{\mu i} + 2p_\mu^\perp p_i}{\alpha\beta_B s + p_\perp^2}|k_\perp)^{ba} + 2ge^{i(k,x)_\perp}\frac{k_\mu^\perp}{k_\perp^2}\tilde{\mathcal{F}}_i^{ba}\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, x_\perp\Big)\\
&\quad -2ge^{i(k,x)_\perp}\Big[\frac{\delta_\mu^j k_i + k_\mu^\perp\delta_i^j - g_{\mu i}k^j}{\alpha\beta_B s + k_\perp^2} + \frac{g_{\mu i}k_\perp^2 k^j + 2k_i k^j k_\mu^\perp}{(\alpha\beta_B s + k_\perp^2)^2}\Big]\breve{\tilde{\mathcal{F}}}_j^{ba}\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, x_\perp\Big) + O(p_{2\mu})
\end{aligned}$$

where

$$\breve{\tilde{\mathcal{F}}}_j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, x_\perp\Big) \equiv \tilde{\mathcal{F}}_j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}, x_\perp\Big) - \tilde{\mathcal{F}}_j(0, x_\perp) \qquad (4.32)$$

Similarly to eq. (4.28) we can rewrite the above expression in the form without subtractions

$$\begin{aligned}
&\tilde{L}_{i\mu}^{ba}(k, x_\perp, \beta_B)^{\text{light-like}} \qquad (4.33)\\
&= g(x_\perp|\Big\{\tilde{U}\frac{1}{\alpha\beta_B s + p_\perp^2}(\tilde{U}^\dagger k_j + p_j\tilde{U}^\dagger)\frac{\alpha\beta_B s g_{\mu i} - 2k_\mu^\perp k_i}{\alpha\beta_B s + k_\perp^2} - 2k_\mu^\perp g_{ij}\tilde{U}\frac{1}{\alpha\beta_B s + p_\perp^2}\tilde{U}^\dagger\\
&\quad -2g_{\mu j}\tilde{U}\frac{p_i}{\alpha\beta_B s + p_\perp^2}\tilde{U}^\dagger + \frac{2k_\mu^\perp}{k_\perp^2}g_{ij}\Big\}\tilde{\mathcal{F}}^j\Big(\beta_B + \frac{k_\perp^2}{\alpha s}\Big)|k_\perp)^{ba} + O(p_{2\mu})
\end{aligned}$$

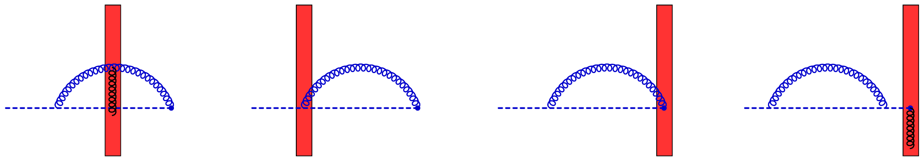
The production part of the evolution kernel is proportional to the cross section of gluon emission given by the product of eqs. (4.26) and (4.31) integrated according to eq. (3.1). To find the full kernel we should calculate the virtual part.

### 4.4 Virtual correction

To get the virtual correction shown in figure 5 we should use the expansion (3.11) of the operator $\mathcal{F}$ up to the second order in quantum field. From eq. (3.11) one gets

$$\begin{aligned}
&\langle\mathcal{F}_i^n(\beta_B, y_\perp)\rangle^{2\text{nd}} \qquad (4.34)\\
&= \frac{2g^2}{s}\int dy_*\, e^{i\beta_B y_*}\Big[\beta_B\int_{y_*}^\infty dz_*\big([\infty, z_*]_y\langle A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, y_*]_y\big)^{nm}A_i^{m\mathrm{q}}(y_*, y_\perp)\rangle\\
&\quad -\frac{2i}{s}\int_{y_*}^\infty dz_*\big([\infty, z_*]_y\langle A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, y_*]_y\big)^{nm}\partial_i A_\bullet^{m\mathrm{q}}(y_*, y_\perp)\rangle\\
&\quad -\frac{4g}{s^2}\int_{y_*}^\infty dz_*\int_{y_*}^{z_*} dz_*'\ \big([\infty, z_*]_y\langle A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, z_*']_y A_\bullet^{\mathrm{q}}(z_*', y_\perp)[z_*', y_*]_y\rangle\big)^{nm}F_{\bullet i}^m(y_*, y_\perp)\Big]
\end{aligned}$$

As in the case of production kernel we will calculate the diagrams in figure 5a, 5b, and 5c separately and then check that the final result does not depend on the size of the shock wave $\sigma_*$ (it is easy to see that the diagram in figure 5d vanishes in Feynman gauge).



(a) (b) (c) (d)

**Figure 5**. Virtual part of the evolution kernel.

### 4.4.1 Diagram in figure 5a

Let us start with the diagram in figure 5a. Using eq. (3.11) and (B.27) we get

$$
\begin{aligned}
&\frac{2}{s}\int_{-\infty}^{-\sigma_*} dy_*\ e^{i\beta_B y_*}\langle[\infty,y_*]_y^{nm} gF_{\bullet i}^m(y_*,y_\perp)\rangle^{\text{figure 5a}} \qquad (4.35)\\
&=\frac{2g^2}{s}\int_{-\infty}^{-\sigma_*} dy_*\ e^{i\beta_B y_*}\Big[\beta_B\int_{\sigma_*}^{\infty} dz_*\big(\langle A_\bullet^{\mathrm{q}}(z_*,y_\perp)U_y\big)^{nm} A_i^{m\mathrm{q}}(y_*,y_\perp)\rangle\\
&\quad -\frac{2i}{s}\int_{\sigma_*}^{\infty} dz_*\big(\langle A_\bullet^{\mathrm{q}}(z_*,y_\perp)U_y\big)^{nm}\partial_i A_\bullet^{m\mathrm{q}}(y_*,y_\perp)\rangle\Big]\\
&=-\frac{ig^2}{s}f^{nkl}\int_0^\infty\frac{{\mathchar'26\mkern-12mu d}\alpha}{\alpha}\int_{-\infty}^{-\sigma_*}dy_*\int_{\sigma_*}^{\infty}dz_*(y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}z_*}\Big\{\beta_B\mathcal{G}_{\bullet i}(\infty,-\infty;p_\perp)\\
&\quad+\frac{2}{s}[\mathcal{G}_{\bullet\bullet}(\infty,-\infty;p_\perp)+\mathcal{Q}_{\bullet\bullet}(\infty,-\infty;p_\perp)]p_i\Big\}e^{i(\beta_B+\frac{p_\perp^2}{\alpha s})y_*}U^\dagger|y_\perp)^{kl}\\
&=ig^2f^{nkl}\int_0^\infty{\mathchar'26\mkern-12mu d}\alpha(y_\perp|\frac{1}{p_\perp^2}e^{-i\frac{p_\perp^2}{\alpha s}\sigma_*}\{\alpha\beta_B s\mathcal{G}_{\bullet i}(\infty,-\infty;p_\perp)\\
&\quad+2\alpha[\mathcal{G}_{\bullet\bullet}(\infty,-\infty;p_\perp)+\mathcal{Q}_{\bullet\bullet}(\infty,-\infty;p_\perp)]p_i\}\frac{1}{\alpha\beta_B s+p_\perp^2}e^{-i(\beta_B+\frac{p_\perp^2}{\alpha s})\sigma_*}U^\dagger|y_\perp)^{kl}
\end{aligned}
$$

(as usual we assume that there are no external fields outside $[\sigma_*,-\sigma_*]$ interval). Moreover, from eq. (B.25) we see that $\mathcal{G}_{\bullet i}(\infty,-\infty;p_\perp)=-\frac{i}{\alpha}\partial_i U$ and from eqs. (B.25), (B.7) and (4.9) that $\mathcal{G}_{\bullet\bullet}(\infty,-\infty;p_\perp)+\mathcal{Q}_{\bullet\bullet}(\infty,-\infty;p_\perp)=-\frac{1}{2\alpha^2}\partial_\perp^2 U$ so we obtain

$$
\begin{aligned}
&\frac{2}{s}g\int_{-\infty}^{-\sigma_*}dy_* e^{i\beta_B y_*}\langle[\infty,y_*]_y^{nm}F_{\bullet i}^m(y_*,y_\perp)\rangle^{\text{figure 5a}} \qquad (4.36)\\
&=g^2f^{nkl}\int_0^\infty\frac{{\mathchar'26\mkern-12mu d}\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}e^{-i\frac{p_\perp^2}{\alpha s}\sigma_*}[\alpha\beta_B s\partial_i U-i\partial_\perp^2 Up_i]\frac{1}{\alpha\beta_B s+p_\perp^2}e^{-i(\beta_B+\frac{p_\perp^2}{\alpha s})\sigma_*}U^\dagger|y_\perp)^{kl}\\
&=g^2f^{nkl}e^{-i\beta_B\sigma_*}\int_0^\infty\frac{{\mathchar'26\mkern-12mu d}\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}[\alpha\beta_B s\partial_i U-i\partial_\perp^2 Up_i]\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger|y_\perp)^{kl}
\end{aligned}
$$

(recall that $\frac{p_\perp^2}{\alpha s}\sigma_*\ll 1$ if the transverse momenta in the loop are of order of transverse momenta of external fields).

#### 4.4.2 Diagram in figure 5b

To get the contribution of the diagram in figure 5b we need the gluon propagator with one point in the shock wave (B.8), which we will rewrite as follows

$$\begin{aligned}\langle A^a_\mu(z_*,z_\perp)A^b_\nu(y_*,y_\perp)\rangle\ &=\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\{(y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(y-z)_*}\\ &\times\big[\mathcal{G}^{ba}_{\nu\mu}(y_*,z_*;p_\perp)+\mathcal{Q}^{ba}_{\nu\mu}(y_*,z_*;p_\perp)\big]|z_\perp)+(z_\perp|\bar{\mathcal{Q}}^{ba}_{\nu\mu}(y_*,z_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(y-z)_*}|y_\perp)\}\end{aligned}\tag{4.37}$$

with $\mathcal{G}$ and $\mathcal{Q}$ given by eqs. (B.6) and (B.7)

$$\begin{aligned}&\mathcal{G}^{ba}_{i\bullet}(y_*,z_*;p_\perp)=\frac{2g}{\alpha s}\int_{y_*}^{z_*}dz'_*([z_*,z'_*]F_{\bullet i}(z'_*)[z'_*,y_*])^{ab};\quad \mathcal{Q}_{i\bullet}(y_*,z_*;p_\perp)=\bar{\mathcal{Q}}_{i\bullet}(y_*,z_*;p_\perp)=0,\\ &\mathcal{G}^{ba}_{\bullet\bullet}(y_*,z_*;p_\perp)=-\frac{4g^2}{\alpha^2s^2}\int_{y_*}^{z_*}dz'_*\int_{y_*}^{z'_*}dz''_*([z_*,z'_*]F_{\bullet j}(z'_*)[z'_*,z''_*]F_\bullet{}^j(z''_*)[z''_*,y_*])^{ab}\\ &\mathcal{Q}^{ba}_{\bullet\bullet}(y_*,z_*;p_\perp)=-\frac{ig}{\alpha^2 s}\int_{y_*}^{z_*}dz'_*([z_*,z'_*]D^jF_{\bullet j}(z'_*)[z'_*,y_*])^{ab},\qquad \bar{\mathcal{Q}}_{\bullet\bullet}(y_*,z_*;p_\perp)=0\end{aligned}\tag{4.38}$$

and therefore from eq. (4.34) we get

$$\begin{aligned}&\frac{2}{s}\int_{-\infty}^{-\sigma_*}dy_*e^{i\beta_By_*}\langle[\infty,y_*]^{nm}_ygF^m_{\bullet i}(y_*,y_\perp)\rangle^{\text{figure 5b}}\\ &=\frac{2g^2}{s}\int_{-\infty}^{-\sigma_*}dy_*e^{i\beta_By_*}\Big[\beta_B\int_{-\sigma_*}^{\sigma_*}dz_*\big([\infty,z_*]_y\langle A^{\mathfrak{q}}_\bullet(z_*,y_\perp)[z_*,y_*]_y\big)^{nm}A^{m\mathfrak{q}}_i(y_*,y_\perp)\rangle\\ &\quad-\frac{2i}{s}\int_{-\sigma_*}^{\sigma_*}dz_*\big([\infty,z_*]_y\langle A^{\mathfrak{q}}_\bullet(z_*,y_\perp)[z_*,y_*]_y\big)^{nm}\partial_iA^{m\mathfrak{q}}_\bullet(y_*,y_\perp)\rangle\Big]\end{aligned}\tag{4.39}$$

First, let us show that the second term in the r.h.s. of this equation vanishes. From eq. (4.37) we see that

$$\begin{aligned}&\langle A^{a\mathfrak{q}}_\bullet(z_*,y_\perp)\partial_iA^{b\mathfrak{q}}_\bullet(y_*,y_\perp)\rangle\\ &=-i\int_{-\infty}^0\frac{đ\alpha}{2\alpha}(y_\perp|p_ie^{-i\frac{p_\perp^2}{\alpha s}(y-z)_*}\big[\mathcal{G}^{ba}_{\bullet\bullet}(y_*,z_*;p_\perp)+\mathcal{Q}^{ba}_{\bullet\bullet}(y_*,z_*;p_\perp)\big]|y_\perp)=0\end{aligned}\tag{4.40}$$

because operators in eq. (4.37) do not contain $p$ and $(y_\perp|p_ie^{-i\frac{p_\perp^2}{\alpha s}(y-z)_*}|y_\perp)=0$.

Now consider the first term in the r.h.s. of eq. (4.39). From eq. (4.37) we get

$$\langle A^a_\bullet(z_*,y_\perp)A^b_i(y_*,y_\perp)\rangle\ =\frac{g}{s}\int_0^\infty\frac{đ\alpha}{\alpha^2}(y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(z-y)_*}\int_{y_*}^{z_*}dz'_*\ [z_*,z'_*]F_{\bullet i}(z'_*)[z'_*,y_*]|y_\perp)^{ab}\tag{4.41}$$

and therefore

$$\begin{aligned}&\frac{2g}{s}\int_{-\infty}^{-\sigma_*}dy_*e^{i\beta_By_*}\langle[\infty,y_*]^{nm}_yF^m_{\bullet i}(y_*,y_\perp)\rangle^{\text{figure 5b}}=2\frac{g^2}{s}if^{nkl}\int_{-\sigma_*}^{\sigma_*}dz'_*\ \mathcal{F}^{kl}_i(z'_*,y_\perp)\\ &\times\int_0^\infty\frac{đ\alpha}{\alpha}(y_\perp|\big[e^{-i\frac{p_\perp^2}{\alpha s}\sigma_*}-e^{-i\frac{p_\perp^2}{\alpha s}z'_*}\big]\frac{\alpha\beta_Bs}{p_\perp^2(\alpha\beta_Bs+p_\perp^2)}e^{-i(\beta_B+\frac{p_\perp^2}{\alpha s})\sigma_*}|y_\perp)\end{aligned}\tag{4.42}$$

where $\mathcal{F}_i^{kl}(z'_*, y_\perp) \equiv g([\infty, z'_*]_y F_{\bullet i}(z'_*, y_\perp)[z'_*, \infty]_y)^{kl}$, see eq. (1.4). Since $\frac{p_\perp^2}{\alpha s}\sigma_* \ll 1$ the r.h.s. of this equation can be simplified to

$$\frac{2g}{s}\int_{-\infty}^{-\sigma_*} dy_* e^{i\beta_B y_*}\langle[\infty, y_*]_y^{nm} F_{\bullet i}^m(y_*, y_\perp)\rangle^{\text{figure 5b}} \tag{4.43}$$
$$= -2\frac{g^2}{s} f^{nkl} e^{-i\beta_B\sigma_*}\int_0^\infty \frac{d\!\!\!{}^-\alpha}{\alpha}(y_\perp|\frac{\beta_B}{\alpha\beta_B s + p_\perp^2}|y_\perp)\int_{-\sigma_*}^{\sigma_*} dz'_* \ (z' - \sigma)_* \mathcal{F}_i^{kl}(z'_*, y_\perp) \ \simeq \ 0$$

because it is $O(\frac{p_\perp^2}{\alpha s}\sigma_*)$ in comparison to eq. (4.36).

### 4.4.3 Diagram in figure 5c

As in previous sections, we start from rewriting eq. (3.11)

$$\frac{2g}{s}\int_{-\sigma_*}^{\sigma_*} dy_* \ e^{i\beta_B y_*}\langle[\infty, y_*]_y^{nm} F_{\bullet i}^m(y_*, y_\perp)\rangle^{\text{figure 5c}} \tag{4.44}$$
$$= \frac{2g^2}{s}\int_{-\sigma_*}^{\sigma_*} dy_* \ e^{i\beta_B y_*}\Big[ -\frac{4g}{s^2}\int_{y_*}^\infty dz_*\int_{y_*}^{z_*} dz'_* \ \big([\infty, z_*]_y\langle A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, z'_*]_y A_\bullet^{\mathrm{q}}(z'_*, y_\perp)\rangle$$
$$\times[z'_*, y_*]_y\big)^{nm} F_{\bullet i}^m(y_*, y_\perp) + \beta_B\int_{y_*}^\infty dz_*\big([\infty, z_*]_y\langle A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, y_*]_y\big)^{nm} A_i^{m\mathrm{q}}(y_*, y_\perp)\rangle$$
$$-\frac{2i}{s}\int_{y_*}^\infty dz_*\big([\infty, z_*]_y\langle A_\bullet^{\mathrm{q}}(z_*, y_\perp)[z_*, y_*]_y\big)^{nm}\partial_i A_\bullet^{m\mathrm{q}}(y_*, y_\perp)\rangle\Big]$$

Using the propagator (B.8) with point $y$ inside the shock wave (and point $z$ anywhere)[6] we obtain (hereafter $\partial_i(\mathcal{C}) \equiv -i[p_i, \mathcal{C}]$)

$$\frac{2g}{s}\int_{-\sigma_*}^{\sigma_*} dy_* \ e^{i\beta_B y_*}\langle[\infty, y_*]_y^{nm} F_{\bullet i}^m(y_*, y_\perp)\rangle^{\text{figure 5c}} \tag{4.45}$$
$$= \frac{g^2}{s} i f^{nkl}\int_{-\sigma_*}^{\sigma_*} dy_* \ e^{i\beta_B y_*}\int_{y_*}^\infty dz_*\int_0^\infty \frac{d\!\!\!{}^-\alpha}{\alpha}\Big\{(y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(z-y)_*}\big[ -\beta_B([\infty, z_*]$$
$$\times\mathcal{G}_{\bullet i}(z_*, y_*; p_\perp)[y_*, \infty])^{kl} + \frac{2i}{s}\{\partial_i\big([\infty, z_*]\mathcal{O}_{\bullet\bullet}(z_*, y_*; p_\perp)[y_*, \infty]\big)$$
$$-\partial_i([\infty, z_*])\mathcal{O}_{\bullet\bullet}(z_*, y_*; p_\perp)[y_*, \infty] - [\infty, z_*]\mathcal{O}_{\bullet\bullet}(z_*, y_*; p_\perp)\partial_i([y_*, \infty])\}^{kl}\big]|y_\perp)$$
$$+\frac{4g}{s^2}\int_{y_*}^{z_*} dz'_* \ (y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(z-z')_*}\big([\infty, z_*]\mathcal{O}_{\bullet\bullet}(z_*, z'_*; p_\perp)[z'_*, y_*]F_{\bullet i}(y_*)[y_*, \infty]\big)^{kl}|y_\perp)\Big\}$$

where $\mathcal{G}_{\bullet i}$ is given by eq. (B.6)

$$\mathcal{G}_{\bullet i}(x_*, y_*; p_\perp) = -\frac{2g}{\alpha s}\int_{y_*}^{x_*} dz_*[x_*, z_*]F_{\bullet i}(z_*)[z_*, y_*] \tag{4.46}$$

and $\mathcal{O}_{\bullet\bullet} \equiv \mathcal{G}_{\bullet\bullet} + \mathcal{Q}_{\bullet\bullet}$ by eqs. (B.6), (B.7)

$$\mathcal{O}_{\bullet\bullet}(x_*, y_*; p_\perp) = \frac{g}{\alpha^2 s}\int_{y_*}^{x_*} dz_* \ [x_*, z_*]\big\{ -iD^j F_{\bullet j}(z_*)[z_*, y_*]$$
$$-\frac{4g}{s}\int_{y_*}^{z_*} dz'_* \ \ F_{\bullet j}(z_*)[z_*, z'_*]F_\bullet{}^j(z'_*)[z'_*, y_*]\big\} \tag{4.47}$$

[6] Strictly speaking, one should depict eq. (4.44) as several diagrams with points $z$ (and $z'$) inside and outside the shock wave.

Using these expressions, one obtains after some algebra

$$
\begin{aligned}
&\frac{2g}{s}\int_{-\sigma_*}^{\sigma_*} dy_*\ e^{i\beta_B y_*}\langle[\infty,y_*]_y^{nm}F_{\bullet i}^m(y_*,y_\perp)\rangle^{\text{figure 5c}} \qquad (4.48)\\
&= g^2 f^{nkl}\int_0^\infty \frac{d\!\!\!^-\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)\Big\{\int_{-\sigma_*}^{\sigma_*} dw_*\ \left(e^{i\beta_B w_*}-e^{-i\beta_B\sigma_*}\right)\\
&\quad\times\Big(-2i\alpha\beta_B\mathcal{F}_i(w_*,y_\perp)-\frac{1}{s}\partial_i^y\mathcal{V}(w_*,y_\perp)+\frac{2i}{s^2}\mathcal{V}(w_*,y_\perp)\int_{w_*}^{\sigma_*} dz'_*\mathcal{F}_i(z'_*,y_\perp)\Big)\\
&\quad+\frac{2ig}{s}\int_{-\sigma_*}^{\sigma_*} dz'_*\int_{z'_*}^{\sigma_*} dw_*\ \Big(\frac{2ig}{s}[\infty,w_*]_y D^j F_{\bullet j}(w_*,y_\perp)[w_*,\infty]_y\big[e^{i\beta_B z'_*}-e^{-i\beta_B\sigma_*}\big]\\
&\quad+\frac{8g^2}{s^2}\int_{z'}^{w_*} dw'_*[\infty,w_*]_y F_{\bullet j}(w_*,y_\perp)[w_*,w'_*]_y F_\bullet{}^j(w'_*)[w'_*,\infty]_y\\
&\quad\times\big[e^{i\beta_B z'}-e^{-i\beta_B\sigma_*}\big]\Big)\mathcal{F}_i(z'_*,y_\perp)\Big\}^{kl}
\end{aligned}
$$

where, as usual, $\mathcal{F}_i^{kl}=g([\infty,w_*]_y F_{\bullet i}(w_*,y_\perp)[w_*,\infty]_y)^{kl}$ and

$$
\begin{aligned}
\mathcal{V}(w_*,y_\perp)\equiv\ &2ig[\infty,w_*]_y D^j F_{\bullet j}(w_*,y_\perp)[w_*,\infty]_y\\
&+\frac{8g^2}{s}\int_{w_*}^{\infty} dw'_*\ [\infty,w'_*]_y F_{\bullet j}(w'_*,y_\perp)[w'_*,w_*]_y F_\bullet{}^j(w_*,y_\perp)[w_*,\infty]_y \qquad (4.49)
\end{aligned}
$$

(cf. eq. (4.14)).

### 4.4.4 The sum of diagrams in figure 5

The total virtual correction coming from figure 5 is given by the sum of eqs. (4.36) and (4.48)

$$
\begin{aligned}
&\frac{2}{s}g\int dy_* e^{i\beta_B y_*}\langle[\infty,y_*]_y^{nm}F_{\bullet i}^m(y_*,y_\perp)\rangle^{\text{figure 5}} \qquad (4.50)\\
&= g^2 f^{nkl}e^{-i\beta_B\sigma_*}\int_0^\infty\frac{d\!\!\!^-\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}[\alpha\beta_B s\partial_i U-i\partial_\perp^2 U p_i]\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger|y_\perp)^{kl}\\
&\quad+g^2 f^{nkl}\int_0^\infty \frac{d\!\!\!^-\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)\Big\{\int_{-\sigma_*}^{\sigma_*} dw_*\ \left(e^{i\beta_B w_*}-e^{-i\beta_B\sigma_*}\right)\\
&\quad\times\Big(-2i\alpha\beta_B\mathcal{F}_i(w_*,y_\perp)-\frac{1}{s}\partial_i^y\mathcal{V}(w_*,y_\perp)+\frac{2i}{s^2}\mathcal{V}(w_*,y_\perp)\int_{w_*}^{\sigma_*} dz'_*\mathcal{F}_i(z'_*,y_\perp)\Big)\\
&\quad+\frac{2ig}{s}\int_{-\sigma_*}^{\sigma_*} dz'_*\int_{z'_*}^{\sigma_*} dw_*\ \Big(\frac{2ig}{s}[\infty,w_*]_y D^j F_{\bullet j}(w_*,y_\perp)[w_*,\infty]_y\big[e^{i\beta_B z'_*}-e^{-i\beta_B\sigma_*}\big]\\
&\quad+\frac{8g^2}{s^2}\int_{z'}^{w_*} dw'_*[\infty,w_*]_y F_{\bullet j}(w_*,y_\perp)[w_*,w'_*]_y F_\bullet{}^j(w'_*)[w'_*,\infty]_y\\
&\quad\times\big[e^{i\beta_B z'_*}-e^{-i\beta_B\sigma_*}\big]\Big)\mathcal{F}_i(z'_*,y_\perp)\Big\}^{kl}
\end{aligned}
$$

Let us prove that with our accuracy it can be approximated as

$$\begin{aligned}&\frac{2}{s}g\int dy_* e^{i\beta_B y_*}\langle[\infty,y_*]_y^{nm}F_{\bullet i}^m(y_*,y_\perp)\rangle^{\text{figure 5}} \\ &= g^2 f^{nkl}\int_0^\infty \frac{\not{d}\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}[\alpha\beta_B s\partial_i U - i\partial_\perp^2 U p_i]\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger|y_\perp)^{kl} \\ &\quad -ig^2 f^{nkl}\int_0^\infty \frac{\not{d}\alpha}{\alpha}(y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)[\mathcal{F}_i(\beta_B,y_\perp)-i\partial_i U_y U_y^\dagger]^{kl}\end{aligned}\tag{4.51}$$

To this end we compare the size of different terms in the r.h.s. of equations (4.50) and (4.51) at $\beta_B\sigma_* \ll 1$ and $\beta_B\sigma_* \geq 1$. In the first case (at $\beta_B\sigma_* \ll 1$) the only surviving terms in the r.h.s.'s of these equations are the first terms and they are obviously equal.

In the second case let us start from eq. (4.51). Since $\beta_B\sigma_* \sim \beta_B\frac{\sigma' s}{p_\perp^2} \geq 1$ we have $\alpha\beta_B s \gg p_\perp^2$ so

$$\text{r.h.s. of eq. (4.51)} = -ig^2 f^{nkl}\int_0^\infty \frac{\not{d}\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}|y_\perp)\mathcal{F}_i^{kl}(\beta_B,y_\perp)\tag{4.52}$$

Let us now compare the size of different terms in the r.h.s. of eq. (4.50). Since $\frac{1}{s}\int dw_*\mathcal{V}(w_*) \sim \partial_\perp^2 UU^\dagger$ the first term in the fourth line $\sim \int dw_*\alpha\beta_B F_{\bullet i}(w_*,y_\perp) \sim \alpha\beta_B s\partial_i U_y U_y^\dagger$ is much greater than the second term $\sim \frac{1}{s}\int dw_*\partial_i\mathcal{V}(w_*,y_\perp) \sim \partial_i\partial_\perp^2 U_y U_y^\dagger$ or the third term $\sim \frac{1}{s^2}\int dw_*\partial_i\mathcal{V}(w_*,y_\perp)\int dw'_* F_{\bullet i}(w'_*) \sim \partial_\perp^2 U_y\partial_i U_y^\dagger$. Moreover, it is easy to see that the terms in the last three lines in eq. (4.50) are of the same order as the terms $\sim\mathcal{V}$ in the fourth line so they are again small in comparison to the term $\sim\mathcal{F}_i$. Thus, we get

$$\begin{aligned}\text{r.h.s. of eq. (4.50)} &= g^2 f^{nkl}e^{-i\beta_B\sigma_*}\int_0^\infty \frac{\not{d}\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}[\partial_i U - \frac{i}{\alpha\beta_B s}\partial_\perp^2 U p_i]U^\dagger|y_\perp)^{kl} \\ &\quad -ig^2 f^{nkl}\int_0^\infty \frac{\not{d}\alpha}{\alpha}(y_\perp|\frac{1}{p_\perp^2}|y_\perp)\frac{2}{s}\int_{-\sigma_*}^{\sigma_*}dw_*\left(e^{i\beta_B w_*}-e^{-i\beta_B\sigma_*}\right)\mathcal{F}_i^{kl}(w_*,y_\perp)\end{aligned}\tag{4.53}$$

which coincides with the r.h.s. of eq. (4.52).

Last but not least, let us prove that one can use the formula (4.51) in the light-cone limit $l_\perp^2 \ll p_\perp^2$ where it coincides with eq. (3.15). First we notice that the term $\sim\partial_\perp^2 U$ has twist two and so exceeds our twist-one light-cone accuracy. Next, since the commutator $[p_\perp^2,\partial_i U]$ consists of operators of collinear twist two (or higher), one can rewrite the first term in the r.h.s. of eq. (4.51) in the form

$$(y_\perp|\frac{1}{p_\perp^2}\alpha\beta_B s\partial_i U\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger|y_\perp) \simeq (y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)\partial_i U_y U_y^\dagger\tag{4.54}$$

so it cancels with last term in the r.h.s. of eq. (4.51) and we obtain

$$\begin{aligned}&\frac{2}{s}g\int dy_* e^{i\beta_B y_*}\langle[\infty,y_*]_y^{nm}F_{\bullet i}^m(y_*,y_\perp)\rangle^{\text{figure 5}} \\ &= -ig^2 f^{nkl}\int_0^\infty \frac{\not{d}\alpha}{\alpha}(y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)\mathcal{F}_i^{kl}(\beta_B,y_\perp)\end{aligned}\tag{4.55}$$

which is the light-cone result eq. (3.15).



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Thus, the final result for the sum of diagrams in figure 5 is eq. (4.51)

$$\langle\mathcal{F}_i^n(\beta_B,y_\perp)\rangle^{\text{figure 5}} = -ig^2f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!{}^-\alpha}{\alpha}\Big\{(y_\perp|\frac{1}{p_\perp^2}[\alpha\beta_B si\partial_i U+\partial_\perp^2 Up_i]\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger|y_\perp)^{kl}$$
$$+(y_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)[\mathcal{F}_i(\beta_B,y_\perp)-U_i(y_\perp)]^{kl}\Big\} \tag{4.56}$$
$$= -ig^2f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!{}^-\alpha}{\alpha}(y_\perp|\frac{p^j}{p_\perp^2}(2\partial_i\partial_j U+g_{ij}\partial_\perp^2 U)\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger+\frac{\alpha\beta_B sp_\perp^{-2}}{\alpha\beta_B s+p_\perp^2}\mathcal{F}_i(\beta_B)|y_\perp)^{kl}$$

where we imposed our cutoff $\sigma>\alpha>\sigma'$. Again, let us note that the above expression is valid with our accuracy in the whole range of transverse momenta.

Similarly to eq. (4.28) we can rewrite this formula in the form without subtractions

$$\langle\mathcal{F}_i^n(\beta_B,y_\perp)\rangle^{\text{figure 5}} = -ig^2f^{nkl}\int_{\sigma'}^{\sigma}\frac{d\!\!\!{}^-\alpha}{\alpha}(y_\perp|-\frac{p^j}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l) \tag{4.57}$$
$$\times(2\delta_j^k\delta_i^l-g_{ij}g^{kl})U\frac{1}{\alpha\beta_B s+p_\perp^2}U^\dagger+\mathcal{F}_i(\beta_B)\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|y_\perp)^{kl}$$

where $\mathcal{F}_k\overleftarrow{\partial}_l\equiv\partial_l\mathcal{F}_k=-i[p_l,\mathcal{F}_k]$. Indeed, in the light-cone case $l_\perp^2\ll p_\perp^2$ one can neglect the operators with high collinear twist so both equations (4.56) and (4.57) reduce to the last terms in the r.h.s's which are the same. Also, as we discussed above, in the shock-wave case ($l_\perp^2\sim p_\perp^2$) and $\beta_B$ small one can replace $\mathcal{F}_i(\beta_B)$ by $U_i$ so the r.h.s's of eq. (4.56) and eq. (4.57) coincide after some trivial algebra. Finally, if $l_\perp^2\sim p_\perp^2$ and $\beta_B\geq\frac{1}{\sigma_*}$ we have $\alpha\beta_B s\gg p_\perp^2$ so again the equations (4.56) and (4.57) reduce to the last terms in the r.h.s.'s.

#### 4.4.5 Virtual correction for the complex conjugate amplitude

The calculation of the virtual correction in the complex conjugate amplitude is very similar so we will only outline it. As in the previous section, we start with the formula (3.16) which can be rewritten as

$$\frac{2}{s}\int dx_*\ e^{-i\beta_B x_*}\langle\tilde{F}_{\bullet i}^m(x_*,x_\perp)[x_*,\infty]_x^{mn}\rangle^{\text{2nd}} \tag{4.58}$$
$$=\frac{2}{s}\int dx_*\ e^{-i\beta_B x_*}\Big\{\beta_B\int_{x_*}^{\infty}dz_*\langle\tilde{A}_i^{m\mathrm{q}}(x_*,x_\perp)\big([x_*,z_*]_x\tilde{A}_\bullet^{\mathrm{q}}(z_*,x_\perp)[z_*,\infty]_x\big)^{mn}\rangle$$
$$+\frac{2i}{s}\int_{x_*}^{\infty}dz_*\langle\partial_i\tilde{A}_\bullet^{m\mathrm{q}}\big([x_*,z_*]_x\tilde{A}_\bullet^{\mathrm{q}}(z_*,x_\perp)[z_*,\infty]_x\big)^{mn}\rangle$$
$$-\frac{4}{s^2}\int_{x_*}^{\infty}dz_*\int_{z_*}^{\infty}dz'_*\ \tilde{F}_{\bullet i}^m(x_*,x_\perp)\big([x_*,z_*]_x\langle\tilde{A}_\bullet^{\mathrm{q}}(z_*,x_\perp)[z_*,z'_*]_x\tilde{A}_\bullet^{\mathrm{q}}(z'_*,x_\perp)\rangle[z'_*,\infty]_x\big)^{mn}\Big\}$$

Using eq. (B.28) we get

$$\frac{2}{s}\int dx_*\ e^{-i\beta_B x_*}\langle\tilde F^m_{\bullet i}(x_*,x_\perp)[x_*,\infty]^{nm}_x\rangle^{\text{figure 5}} \tag{4.59}$$
$$= -\frac{ig^2}{s}f^{nkl}\int_0^\infty\frac{đ\alpha}{\alpha}\int_{-\infty}^{-\sigma_*}dx_*\int_{\sigma_*}^\infty dz_*(x_\perp|\tilde U e^{-i(\beta_B+\frac{p_\perp^2}{\alpha s})x_*}\Big\{\beta_B\tilde{\mathcal G}_{i\bullet}(-\infty,\infty;p_\perp)$$
$$+\frac{2}{s}p_i[\tilde{\mathcal G}_{\bullet\bullet}(-\infty,\infty;p_\perp)+\tilde{\mathcal Q}_{\bullet\bullet}(-\infty,\infty;p_\perp)]\Big\}e^{i\frac{p_\perp^2}{\alpha s}z_*}|x_\perp)^{kl}$$
$$+\frac{g^2}{s}if^{nkl}\int_{-\sigma_*}^{\sigma_*}dx_*\ e^{-i\beta_B x_*}\int_{x_*}^\infty dz_*\int_0^\infty\frac{đ\alpha}{\alpha}\Big\{(x_\perp|\big[-\beta_B([\infty,x_*]\tilde{\mathcal G}_{i\bullet}(x_*,z_*;p_\perp)[z_*,\infty])^{kl}$$
$$-\frac{2i}{s}\big\{\partial_i\big([\infty,x_*]\tilde{\mathcal O}_{\bullet\bullet}(x_*,z_*;p_\perp)[z_*,\infty]\big)-\partial_i([\infty,x_*])\tilde{\mathcal O}_{\bullet\bullet}(x_*,z_*;p_\perp)[z_*,\infty]$$
$$-[\infty,x_*]\tilde{\mathcal O}_{\bullet\bullet}(x_*,z_*;p_\perp)\partial_i([z_*,\infty])\big\}^{kl}\big]e^{-i\frac{p_\perp^2}{\alpha s}(x-z)_*}|x_\perp)$$
$$+\frac{4}{s^2}\int_{z_*}^\infty dz'_*\ (x_\perp|\big([\infty,x_*]\tilde F_{\bullet i}(x_*)[x_*,z_*]\tilde{\mathcal O}_{\bullet\bullet}(z_*,z'_*;p_\perp)[z'_*,\infty]\big)^{kl}e^{-i\frac{p_\perp^2}{\alpha s}(z-z')_*}|x_\perp)\Big\}$$

Similarly to eq. (4.50) it is possible to demonstrate that the last three lines in the r.h.s. of this equation exceed our accuracy, and moreover, one can neglect factors $e^{-i\beta_B\sigma_*}$. Using formulas (B.29) for $\tilde{\mathcal G}_{\mu\nu}$ and (B.10) for $\tilde{\mathcal Q}_{\mu\nu}$ we obtain the virtual correction in the complex conjugate amplitude in the form

$$\langle\tilde{\mathcal F}^n_i(\beta_B,x_\perp)\rangle^\sigma=\frac{2g}{s}\int dx_*\ e^{-i\beta_B x_*}\langle\tilde F^m_{\bullet i}(x_*,x_\perp)[x_*,\infty]^{mn}_x\rangle^\sigma$$
$$=-ig^2f^{nkl}\int_{\sigma'}^\sigma\frac{đ\alpha}{\alpha}\Big\{(x_\perp|\tilde U\frac{1}{\alpha\beta_B s+p_\perp^2}\big\{-i\alpha\beta_B s\partial_i\tilde U^\dagger+p_i\partial_\perp^2\tilde U^\dagger\big\}\frac{1}{p_\perp^2}|x_\perp)^{kl}$$
$$+\big(\tilde{\mathcal F}_i(\beta_B,x_\perp)-i\partial_i\tilde U_x\tilde U^\dagger_x\big)^{kl}(x_\perp|\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|x_\perp)\Big\} \tag{4.60}$$

where we have imposed our cutoffs in $\alpha$ and used the formula

$$\partial_\perp^2U^\dagger_z=g\int_{-\infty}^\infty dz_*\ [-\infty,z_*]_z$$
$$\times\Big(-\frac{2i}{s}D^jF_{\bullet j}(z_*,z_\perp)[z_*,\infty]_z+\frac{8g}{s^2}\int_{z_*}^\infty dz'_*\ F_{\bullet j}(z_*,z_\perp)[z_*,z'_*]_zF_\bullet{}^j(z'_*,z_\perp)[z'_*,\infty]_z\Big)$$

Similarly to eq. (4.51) this expression is also valid in the light-cone case $l_\perp^2\ll p_\perp^2$ where it coincides with eq. (3.18).

Alternatively, one can use the expression without subtractions (cf. eq. (4.57))

$$\langle\tilde{\mathcal F}^n_i(\beta_B,x_\perp)\rangle^\sigma=-ig^2f^{nkl}\int_{\sigma'}^\sigma\frac{đ\alpha}{\alpha}(x_\perp|\tilde U\frac{1}{\alpha\beta_B s+p_\perp^2}\tilde U^\dagger \tag{4.61}$$
$$\times(2\delta_i^k\delta_j^l-g_{ij}g^{kl})(i\partial_k-\tilde U_k)\tilde{\mathcal F}_l(\beta_B)\frac{p^j}{p_\perp^2}+\tilde{\mathcal F}_i(\beta_B)\frac{\alpha\beta_B s}{p_\perp^2(\alpha\beta_B s+p_\perp^2)}|x_\perp)^{kl}$$

## 5 Evolution equation for gluon TMD

Now we are in a position to assemble all leading-order contributions to the rapidity evolution of gluon TMD. Adding the production part (3.1) with Lipatov vertices (4.28) and (4.33) and the virtual parts from previous section (4.57) and (4.61) we obtain

$$\begin{aligned}
&\big(\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp)\big)^{\ln\sigma} \qquad (5.1)\\
&= -g^2\int_{\sigma'}^{\sigma}\frac{d\!\!\!{}^-\alpha}{2\alpha}\int d\!\!\!{}^-{}^2k_\perp\ \mathrm{Tr}\{\tilde{L}_i{}^{\mu}(k,x_\perp,\beta_B)^{\mathrm{light-like}}L_{\mu j}(k,y_\perp,\beta_B)^{\mathrm{light-like}}\}\\
&\quad -g^2\int_{\sigma'}^{\sigma}\frac{d\!\!\!{}^-\alpha}{\alpha}\mathrm{Tr}\Big\{\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\alpha\beta_Bs+p_\perp^2}U^\dagger\\
&\quad +\mathcal{F}_j(\beta_B)\frac{\alpha\beta_Bs}{p_\perp^2(\alpha\beta_Bs+p_\perp^2)}|y_\perp)\\
&\quad +(x_\perp|\tilde{U}\frac{1}{\alpha\beta_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}\\
&\quad +\tilde{\mathcal{F}}_i(\beta_B)\frac{\alpha\beta_Bs}{p_\perp^2(\alpha\beta_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\}+O(\alpha_s^2)
\end{aligned}$$

where Tr is a trace in the adjoint representation. In the explicit form the evolution equation reads

$$\begin{aligned}
&\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp) \qquad (5.2)\\
&= -\alpha_s\mathrm{Tr}\Big\{\int d\!\!\!{}^-{}^2k_\perp(x_\perp|\Big\{\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}(\tilde{U}^\dagger k_k+p_k\tilde{U}^\dagger)\frac{\sigma\beta_Bsg_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_Bs+k_\perp^2}\\
&\quad -2k^\perp_\mu g_{ik}\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger-2g_{\mu k}\tilde{U}\frac{p_i}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger+\frac{2k^\perp_\mu}{k_\perp^2}g_{ik}\Big\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad \times(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big\{\frac{\sigma\beta_Bs\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_Bs+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger\\
&\quad -2k^\mu_\perp g_{jl}U\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger-2\delta^\mu_lU\frac{p_j}{\sigma\beta_Bs+p_\perp^2}U^\dagger+2g_{jl}\frac{k^\mu_\perp}{k_\perp^2}\Big\}|y_\perp)\\
&\quad +2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger\\
&\quad +\mathcal{F}_j(\beta_B)\frac{\sigma\beta_Bs}{p_\perp^2(\sigma\beta_Bs+p_\perp^2)}|y_\perp)\\
&\quad +2(x_\perp|\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}\\
&\quad +\tilde{\mathcal{F}}_i(\beta_B)\frac{\sigma\beta_Bs}{p_\perp^2(\sigma\beta_Bs+p_\perp^2)}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big\}+O(\sigma_s^2)
\end{aligned}$$

The operators $\tilde{\mathcal{F}}_j(\beta)$ and $\mathcal{F}_i(\beta)$ are defined as usual, see eq. (4.29)

$$\begin{aligned}
(x_\perp|\tilde{\mathcal{F}}_i(\beta)|k_\perp) &= \frac{2}{s}\int dx_*\ \tilde{\mathcal{F}}_i(x_*,x_\perp)e^{-i\beta x_*+i(k,x)_\perp}\\
(k_\perp|\mathcal{F}_i(\beta)|y_\perp) &= \frac{2}{s}\int dy_*\ e^{i\beta y_*-i(k,y)_\perp}\mathcal{F}_i(y_*,y_\perp) \qquad (5.3)
\end{aligned}$$

The evolution equation (5.2) can be rewritten in the form where cancellation of IR and

UV divergencies is evident

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp) \qquad\qquad (5.4)\\
&= -\alpha_s\mathrm{Tr}\bigg\{\int d\!\!\!^-{}^2k_\perp(x_\perp|\bigg\{\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}(\tilde{U}^\dagger k_k+p_k\tilde{U}^\dagger)\frac{\sigma\beta_Bsg_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_Bs+k_\perp^2}\\
&\quad -2k^\perp_\mu g_{ik}\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger-2g_{\mu k}\tilde{U}\frac{p_i}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger\bigg\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad\times(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\bigg\{\frac{\sigma\beta_Bs\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_Bs+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger\\
&\quad -2k^\mu_\perp g_{jl}U\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger-2\delta^\mu_lU\frac{p_j}{\sigma\beta_Bs+p_\perp^2}U^\dagger\bigg\}|y_\perp)+2\int d\!\!\!^-{}^2k_\perp(x_\perp|\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad\times(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\bigg\{\frac{k_j}{k_\perp^2}\frac{\sigma\beta_Bs+2k_\perp^2}{\sigma\beta_Bs+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger\\
&\quad +2U\frac{g_{jl}}{\sigma\beta_Bs+p_\perp^2}U^\dagger-2\frac{k_l}{k_\perp^2}U\frac{p_j}{\sigma\beta_Bs+p_\perp^2}U^\dagger\bigg\}|y_\perp)\\
&\quad +2\int d\!\!\!^-{}^2k_\perp(x_\perp|\bigg\{\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}(\tilde{U}^\dagger k_k+p_k\tilde{U}^\dagger)\frac{k_i}{k_\perp^2}\frac{\sigma\beta_Bs+2k_\perp^2}{\sigma\beta_Bs+k_\perp^2}+2\tilde{U}\frac{g_{ik}}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger\\
&\quad -2\tilde{U}\frac{p_i}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger\frac{k_k}{k_\perp^2}\bigg\}\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|y_\perp)\\
&\quad +2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger|y_\perp)\\
&\quad +2(x_\perp|\tilde{U}\frac{1}{\sigma\beta_Bs+p_\perp^2}\tilde{U}^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-\tilde{U}_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\\
&\quad -4\int\frac{d\!\!\!^-{}^2k_\perp}{k_\perp^2}\bigg[\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp\Big)e^{i(k,x-y)_\perp}\\
&\quad -\frac{\sigma\beta_Bs}{\sigma\beta_Bs+k_\perp^2}\tilde{\mathcal{F}}_i(\beta_B,x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\bigg]\bigg\}+O(\alpha_s^2).
\end{aligned}
$$

The evolution equation (5.4) is one of the main results of this paper. It describes the rapidity evolution of the operator (2.8) at any Bjorken $x_B\equiv\beta_B$ and any transverse momenta.

Let us discuss the gauge invariance of this equation. The l.h.s. is gauge invariant after taking into account gauge link at $+\infty$ as shown in eq. (2.7). As to the right side, it was obtained by calculation in the background field and promoting the background fields to operators in a usual way. However, we performed our calculations in a specific background field $A_\bullet(x_*,x_\perp)$ with a finite support in $x_\perp$ and we need to address the question how can we restore the r.h.s. of eq. (5.4) in a generic field $A_\mu$. It is easy to see how one can restore the gauge-invariant form: just add gauge link at $+\infty p_1$ or $-\infty p_1$ appropriately. For example, the terms $U_z(z|\frac{1}{\sigma\beta s+p_\perp^2}|z')U^\dagger_{z'}$ in r.h.s. of should be replaced by $U_z[z_\perp-\infty p_1,z'_\perp-\infty p_1](z|\frac{1}{\sigma\beta s+p_\perp^2}|z')U^\dagger_{z'}$. After performing these insertions we will have the result which is (i) gauge invariant and (ii) coincides with eq. (5.4) for our choice of background field. At this step, the background fields in the r.h.s. of eq. (5.4) can be promoted to operators. However, the explicit display of these gauge links at $\pm\infty$ will make the evolution equation much less readable so we will assume they are always in place rather than written explicitly.

When we consider the evolution of gluon TMD (1.6) given by the matrix element (2.4) of the operator (2.8) we need to take into account the kinematical constraint $k_\perp^2 \leq \alpha(1-\beta_B)s$ in the production part of the amplitude. Indeed, as we discussed in section 3.3, the initial hadron's momentum is $p \simeq p_2$ so the sum of the fraction $\beta_B p_2$ and the fraction $\frac{k_\perp^2}{\alpha s}p_2$ carried by the emitted gluon should be smaller than $p_2$. We obtain ($\eta \equiv \ln\sigma$)[7]

$$
\begin{aligned}
&\frac{d}{d\eta}\langle p|\tilde{\mathcal{F}}^a_i(\beta_B,x_\perp)\mathcal{F}^a_j(\beta_B,y_\perp)|p\rangle^\eta \qquad (5.5)\\
&= -\alpha_s\langle p|\mathrm{Tr}\Big\{\int {\mathchar'26\mkern-12mu d}^2k_\perp\,\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\Big[(x_\perp|\Big(U\frac{1}{\sigma\beta_B s+p_\perp^2}(U^\dagger k_k+p_kU^\dagger)\\
&\quad\times\frac{\sigma\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_B s+k_\perp^2}-2k^\perp_\mu g_{ik}U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger-2g_{\mu k}U\frac{p_i}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad\times(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{\sigma\beta_B s\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_B s+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&\quad-2k^\mu_\perp g_{jl}U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger-2\delta^\mu_l U\frac{p_j}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)|y_\perp)\\
&\quad+2(x_\perp|\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{k_j}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}(k_lU+Up_l)\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&\quad+2U\frac{g_{jl}}{\sigma\beta_B s+p_\perp^2}U^\dagger-2\frac{k_l}{k_\perp^2}U\frac{p_j}{\sigma\beta_B s+p_\perp^2}U^\dagger\Big)|y_\perp)\\
&\quad+2(x_\perp|\Big(U\frac{1}{\sigma\beta_B s+p_\perp^2}(U^\dagger k_k+p_kU^\dagger)\frac{k_i}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}+2U\frac{g_{ik}}{\sigma\beta_B s+p_\perp^2}U^\dagger\\
&\quad-2U\frac{p_i}{\sigma\beta_B s+p_\perp^2}U^\dagger\frac{k_k}{k_\perp^2}\Big)\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s}\Big)|y_\perp)\Big]\\
&\quad+2\tilde{\mathcal{F}}_i(\beta_B,x_\perp)(y_\perp|-\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l+U_l)(2\delta^k_m\delta^l_j-g_{jm}g^{kl})U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger|y_\perp)\\
&\quad+2(x_\perp|U\frac{1}{\sigma\beta_B s+p_\perp^2}U^\dagger(2\delta^k_i\delta^l_m-g_{im}g^{kl})(i\partial_k-U_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\\
&\quad-4\int\frac{{\mathchar'26\mkern-12mu d}^2k_\perp}{k_\perp^2}\Big[\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s},y_\perp\Big)e^{i(k,x-y)_\perp}\\
&\quad-\frac{\sigma\beta_B s}{\sigma\beta_B s+k_\perp^2}\tilde{\mathcal{F}}_i(\beta_B,x_\perp)\mathcal{F}_j(\beta_B,y_\perp)\Big]\Big\}|p\rangle^\eta+O(\alpha_s^2)
\end{aligned}
$$

Note that we erased tilde from Wilson lines since we have a sum over full set of states and gluon operators at space-like (or light-like) intervals commute with each other.[8] This equation describes the rapidity evolution of gluon TMD (1.6) with rapidity cutoff (2.1) in

[7] Strictly speaking, we need to consider matrix element $\langle p|\tilde{\mathcal{F}}^{ai}(\beta_B,x_\perp)\mathcal{F}^a_i(\beta_B,y_\perp)|p+\xi p_2\rangle$ proportional to $\delta(\xi)$, see eq. (2.4).

[8] We have left $\tilde{F}$ as a reminder of different signs in the exponents of Fourier transforms in the definitions (2.2) and (2.3).



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the whole range of $\beta_B = x_B$ and $k_\perp$ ($\sim |x-y|_\perp^{-1}$). In the next section we will consider some specific cases.

## 6 BK, DGLAP, and Sudakov limits of TMD evolution equation

### 6.1 Small-x case: BK evolution of the Weizsacker-Williams distribution

First, let us consider the evolution of Weizsacker-Williams (WW) unintegrated gluon distribution (1.1) which can be obtained from eq. (5.5) by setting $\beta_B = 0$. Moreover, in the small-$x$ regime it is assumed that the energy is much higher than anything else so the characteristic transverse momenta $p_\perp^2 \sim (x-y)_\perp^{-2} \ll s$ and in the whole range of evolution ($1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$) we have $\frac{p_\perp^2}{\sigma s} \ll 1$, hence the kinematical constraint $\theta\big(1-\beta_B-\frac{k_\perp^2}{\alpha s}\big)$ in eq. (5.5) can be omitted. Under these assumptions, all $\mathcal{F}_i\big(\beta_B+\frac{p_\perp^2}{\sigma s}\big)$ and $\mathcal{F}_i(\beta_B)$ can be replaced by $i\partial_i UU^\dagger$ and similarly for the complex conjugate amplitude. To simplify algebra, it is convenient to take the production part of the kernel in the form of product of Lipatov vertices (4.26) and (4.31) noting that the "subtraction terms" $\check{\tilde{\mathcal{F}}}_i$ and $\check{\mathcal{F}}_j$ vanish in this limit. One obtains the rapidity evolution of the WW distribution in the form

$$\begin{aligned}
\frac{d}{d\ln\sigma}\tilde{U}_i^a(x_\perp)U_j^a(y_\perp) = -4\alpha_s\mathrm{Tr}\Big\{&(x_\perp|\tilde{U}p_i\tilde{U}^\dagger\Big(\tilde{U}\frac{p^k}{p_\perp^2}\tilde{U}^\dagger-\frac{p^k}{p_\perp^2}\Big)\Big(U\frac{p_k}{p_\perp^2}U^\dagger-\frac{p_k}{p_\perp^2}\Big)Up_jU^\dagger|y_\perp)\\
&-\Big[(x_\perp|\tilde{U}\frac{p_ip^k}{p_\perp^2}\tilde{U}^\dagger\frac{p_k}{p_\perp^2}|x_\perp)-\frac{1}{2}(x_\perp|\frac{1}{p_\perp^2}|x_\perp)\tilde{U}_i(x_\perp)\Big]U_j(y_\perp)\\
&-\tilde{U}_i(x_\perp)\Big[(y_\perp|\frac{p^k}{p_\perp^2}U\frac{p_jp_k}{p_\perp^2}U^\dagger|y_\perp)-\frac{1}{2}(y_\perp|\frac{1}{p_\perp^2}|y_\perp)U_j(y_\perp)\Big]\Big\}
\end{aligned}\tag{6.1}$$

where we used the formula

$$\begin{aligned}
&-(y_\perp|\frac{p^k}{p_\perp^2}(2\partial_j\partial_kU+g_{jk}\partial_\perp^2U)\frac{1}{\sigma\beta_Bs+p_\perp^2}U^\dagger|y_\perp)\\
&=(y_\perp|\frac{p^k}{p_\perp^2}U\frac{p_\perp^2g_{jk}+2p_jp_k}{\sigma\beta_Bs+p_\perp^2}U^\dagger-\frac{1}{\sigma\beta_Bs+p_\perp^2}U_j|y_\perp)
\end{aligned}\tag{6.2}$$

In this form eq. (6.1) agrees with the results of ref. [25]. To see the relation to the BK equation it is convenient to rewrite eq. (6.1) as follows [35] (cf. ref. [36]):

$$\begin{aligned}
&\frac{d}{d\eta}\tilde{U}_i^a(z_1)U_j^a(z_2)\\
&=-\frac{g^2}{8\pi^3}\mathrm{Tr}\Big\{(-i\partial_i^{z_1}+\tilde{U}_i^{z_1})\Big[\int d^2z_3(\tilde{U}_{z_1}\tilde{U}_{z_3}^\dagger-1)\frac{z_{12}^2}{z_{13}^2z_{23}^2}(U_{z_3}U_{z_2}^\dagger-1)\Big](i\overleftarrow{\partial}_j^{z_2}+U_j^{z_2})\Big\}
\end{aligned}\tag{6.3}$$

where $\eta\equiv\ln\sigma$ as usual. In this equation all indices are 2-dimensional and Tr stands for the trace in the adjoint representation. It is easy to see that the expression in the square brackets is actually the BK kernel for the double-functional integral for cross sections [25, 37]. Hereafter, to ensure gauge invariance, $U_i(z_\perp)$ must be understood as $U_i(z_\perp)\equiv\mathcal{F}_i(0,z_\perp)=\frac{2}{s}\int dz_*\ [\infty,z_*]_zF_{\bullet i}(z_*,z_\perp)[z_*,\infty]$ and gauge links at $\infty p_1$ must be inserted as discussed after eq. (5.4).

It is worth noting that eq. (6.3) hold true also at small $\beta_B$ up to $\beta_B \sim \frac{(x-y)_\perp^{-2}}{s}$ since in the whole range of evolution $1 \gg \sigma \gg \frac{(x-y)_\perp^{-2}}{s}$ one can neglect $\sigma\beta_B s$ in comparison to $p_\perp^2$ in eq. (5.5). This effectively reduces $\beta_B$ to 0 so one reproduces eq. (6.3).

### 6.2 Large transverse momenta and the light-cone limit

Now let us discuss the case when $\beta_B = x_B \sim 1$ and $(x-y)_\perp^{-2} \sim s$. At the start of the evolution (at $\sigma \sim 1$) the cutoff in $p_\perp^2$ in the integrals of eq. (5.4) is $\sim (x-y)_\perp^{-2}$. However, as the evolution in rapidity ($\sim \ln\sigma$) progresses the characteristic $p_\perp^2$ becomes smaller due to the kinematical constraint $p_\perp^2 < \sigma(1-\beta_B)s$. Due to this kinematical constraint evolution in $\sigma$ is correlated with the evolution in $p_\perp^2$: if $\sigma \gg \sigma'$ the corresponding transverse momenta of background fields ${p'_\perp}^2$ are much smaller than $p_\perp^2$ in quantum loops. This means that during the evolution we are always in the light-cone case considered in section 3 and therefore the evolution equation for $\beta_B = x_B \sim 1$ and $(x-y)_\perp^{-2} \sim s$ is eq. (3.25) which reduces to the system of evolution equations for gluon TMDs $\mathcal{D}(\beta_B, |z_\perp|, \ln\sigma)$ and $\mathcal{H}(\beta_B, |z_\perp|, \ln\sigma)$ in the case of unpolarized hadron.

### 6.3 Sudakov logarithms

Finally, let us consider the evolution of $\mathcal{D}(x_B, k_\perp, \eta = \ln\sigma)$ in the region where $x_B \equiv \beta_B \sim 1$ and $k_\perp^2 \sim (x-y)_\perp^{-2} \sim$ few GeV$^2$. In this case the integrals over $p_\perp^2$ in the production part of the kernel (5.5) are $\sim (x-y)_\perp^{-2} \sim k_\perp^2$ so that $p_\perp^2 \ll \sigma\beta_B s$ for the whole range of evolution $1 > \sigma > \frac{k_\perp^2}{s}$. For the same reason, the kinematical constraint $\theta\big(1-\beta_B-\frac{p_\perp^2}{\sigma s}\big)$ in the last line of eq. (5.5) can be omitted and we get

$$\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)|p\rangle^{\rm real} \\
&= 4\alpha_s N_c \int \frac{d\!\!{}^-{}^2 p_\perp}{p_\perp^2} e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(\beta_B + \frac{p_\perp^2}{\sigma s}, x_\perp\Big)\mathcal{F}_j^a\Big(\beta_B + \frac{p_\perp^2}{\sigma s}, y_\perp\Big)|p\rangle
\end{aligned} \tag{6.4}$$

As to the virtual part

$$\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)|p\rangle^{\rm virtual} \\
&= 4\alpha_s N_c \int \frac{d\!\!{}^-{}^2 p_\perp}{p_\perp^2}\Big[-\frac{\sigma\beta_B s}{\sigma\beta_B s + p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B, x_\perp)\mathcal{F}_j^a(\beta_B, y_\perp)|p\rangle\Big] \\
&\quad -2\alpha_s {\rm Tr}\langle p|(x_\perp|\tilde{U}\frac{1}{\sigma\beta_B s + p_\perp^2}\tilde{U}^\dagger(2\delta_i^k\delta_m^l - g_{im}g^{kl})(i\partial_k - \tilde{U}_k)\tilde{\mathcal{F}}_l(\beta_B)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{F}_j(\beta_B, y_\perp) \\
&\quad -\tilde{\mathcal{F}}_i(\beta_B, x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l + U_l)(2\delta_m^k\delta_j^l - g_{jm}g^{kl})U\frac{1}{\sigma\beta_B s + p_\perp^2}U^\dagger|y_\perp)|p\rangle
\end{aligned} \tag{6.5}$$

the two last lines can be omitted. Indeed, as we saw in the end of section 4.4.4, these terms are non-vanishing only for the region of large $p_\perp^2 \sim \sigma\beta_B s$. In this region one can expand the operator $\mathcal{O} \equiv \mathcal{F}_k(\beta_B)(i\overleftarrow{\partial}_l + U_l)(2\delta_m^k\delta_j^l - g_{jm}g^{kl})U$ as $\mathcal{O}(z_\perp) = \mathcal{O}(y_\perp) + (y-z)_i\partial_i\mathcal{O}(y_\perp) + \ldots$ and get

$$(y_\perp|\frac{p_m}{p_\perp^2}\mathcal{O}\frac{1}{\sigma\beta_B s + p_\perp^2}|y_\perp) = \mathcal{O}_y(y_\perp|\frac{p_m}{p_\perp^2(\sigma\beta_B s + p_\perp^2)}|y_\perp) + \frac{i\partial_m\mathcal{O}_y}{4\pi\sigma\beta_B s} + \ldots$$

The first term in the r.h.s. of this equation is obviously zero while the second is $O(\frac{m_N^2}{\sigma\beta_B s})$ in comparison to the leading first term in the r.h.s. of eq. (6.5) (the transverse momenta inside the hadron target are $\sim m_N \sim 1\text{GeV}$).

Thus, we obtain the following rapidity evolution equation in the Sudakov region:

$$\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle \qquad (6.6)\\
&= 4\alpha_s N_c\int\frac{d\!\!^-{}^2p_\perp}{p_\perp^2}\Big[e^{i(p,x-y)_\perp}\langle p|\tilde{\mathcal{F}}_i^a\Big(\beta_B+\frac{p_\perp^2}{\sigma s},x_\perp\Big)\mathcal{F}_j^a\Big(\beta_B+\frac{p_\perp^2}{\sigma s},y_\perp\Big)|p\rangle\\
&\quad -\frac{\sigma\beta_B s}{\sigma\beta_B s+p_\perp^2}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle\Big]
\end{aligned}$$

As we mentioned above, the integrals over $p_\perp^2$ in the production part of the kernel (6.6) are $k_\perp^2$ whereas in the virtual part the logarithmic integrals over $p_\perp^2$ are restricted from above by an extra $\frac{1}{p_\perp^2+\sigma\beta_B s}$ leading to the double-log region where $1\gg\sigma\gg\frac{(x-y)_\perp^{-2}}{s}$ and $\sigma\beta_B s\gg p_\perp^2\gg(x-y)_\perp^{-2}$. In that region only the first term in the r.h.s. of eq. (6.6) survives so the evolution equation reduces to

$$\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle^{\eta=\ln\sigma} \qquad (6.7)\\
&= -\frac{g^2N_c}{\pi}\int\frac{d\!\!^-{}^2p_\perp}{p_\perp^2}\big[1-e^{i(p,x-y)_\perp}\big]\langle p|\tilde{\mathcal{F}}_i^a(\beta_B,x_\perp)\mathcal{F}_j^a(\beta_B,y_\perp)|p\rangle^{\eta}
\end{aligned}$$

which can be rewritten for the TMD (1.6) as

$$\frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}(x_B,z_\perp,\ln\sigma)\int\frac{d^2p_\perp}{p_\perp^2}\big[1-e^{i(p,z)_\perp}\big] \qquad (6.8)$$

We see that the IR divergence at $p_\perp^2\to 0$ cancels while the UV divergence in the virtual correction should be cut from above by the condition $p_\perp^2\leq\sigma s$ following from eq. (6.6). With the double-log accuracy one obtains

$$\frac{d}{d\ln\sigma}\mathcal{D}(x_B,z_\perp,\ln\sigma) = -\frac{\alpha_s N_c}{\pi}\mathcal{D}(x_B,z_\perp,\ln\sigma)\ln\sigma s z_\perp^2+\dots \qquad (6.9)$$

where dots stand for the non-logarithmic contributions. This equation leads to the usual Sudakov double-log result

$$\mathcal{D}(x_B,k_\perp,\ln\sigma) \ \sim\ \exp\Big\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma s}{k_\perp^2}\Big\}\mathcal{D}\Big(x_B,k_\perp,\ln\frac{k_\perp^2}{s}\Big) \qquad (6.10)$$

It is worth noting that the coefficient in front of $\ln^2\frac{\sigma s}{k_\perp^2}$ is determined by the cusp anomalous dimension of two light-like Wilson lines going from point $y$ to $\infty p_1$ and $\infty p_2$ directions (with our cutoff $\alpha<\sigma$). Indeed, if one calculates the contribution of the diagram in figure 6 for Wilson lines in the adjoint representation, one gets

$$\begin{aligned}
\langle[\infty p_1,0][0,\infty p_2]\rangle &= -ig^2N_c\int d\!\!^-\alpha\, d\!\!^-\beta\, d\!\!^-{}^2p_\perp\,\frac{\theta(\sigma>|\alpha|>\sigma')}{\alpha(\beta-i\epsilon)(\alpha\beta s-p_\perp^2+i\epsilon)}\\
&= -g^2N_c\int_{\sigma'}^{\sigma}\frac{d\!\!^-\alpha}{\alpha}\int\frac{d\!\!^-{}^2p_\perp}{p_\perp^2} = -\frac{g^2}{8\pi^2}N_c\ln\frac{\sigma}{\sigma'}\int\frac{dp_\perp^2}{p_\perp^2} \qquad (6.11)
\end{aligned}$$

which coincides with the coefficient in eq. (6.8), cf ref. [38–40].

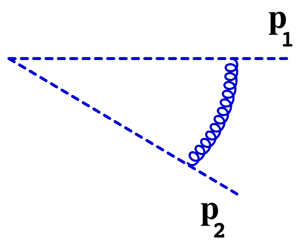


**Figure 6**. Cusp anomalous dimension in the leading order.

## 7 Rapidity evolution of unintegrated gluon distribution in linear approximation

It is instructive to present the evolution kernel (5.5) in the linear (two-gluon) approximation. Since in the r.h.s. of eq. (5.5) we already have $\tilde{\mathcal{F}}_k$ and $\mathcal{F}_l$ (and each of them has at least one gluon) all factors $U$ and $\tilde{U}$ in the r.h.s. of eq. (5.5) can be omitted and we get

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle p|\tilde{\mathcal{F}}^a_i(\beta_B,p_\perp)\mathcal{F}^a_j(\beta_B,p'_\perp)|p\rangle \qquad (7.1)\\
&= -\alpha_s N_c\int d\!\!{}^-{}^2k_\perp\bigg\{\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\bigg[\Big(\frac{(p+k)_k}{\sigma\beta_B s+p_\perp^2}\frac{\sigma\beta_B s g_{\mu i}-2k^\perp_\mu k_i}{\sigma\beta_B s+k_\perp^2}-2\frac{k^\perp_\mu g_{ik}+p_i g_{\mu k}}{\sigma\beta_B s+p_\perp^2}\Big)\\
&\quad\times\Big(\frac{\sigma\beta_B s\delta^\mu_j-2k^\mu_\perp k_j}{\sigma\beta_B s+k_\perp^2}\frac{(p'+k)_l}{\sigma\beta_B s+p'^2_\perp}-2\frac{k^\mu_\perp g_{jl}+\delta^\mu_l p'_j}{\sigma\beta_B s+p'^2_\perp}\Big)\\
&\quad+2g_{ik}\Big(\frac{k_j}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}\frac{(p'+k)_l}{\sigma\beta_B s+p'^2_\perp}+\frac{2g_{jl}}{\sigma\beta_B s+p'^2_\perp}-\frac{2p'_jk_l}{k_\perp^2(\sigma\beta_B s+p'^2_\perp)}\Big)\\
&\quad+2g_{lj}\Big(\frac{(p+k)_k}{\sigma\beta_B s+p_\perp^2}\frac{k_i}{k_\perp^2}\frac{\sigma\beta_B s+2k_\perp^2}{\sigma\beta_B s+k_\perp^2}+\frac{2g_{ik}}{\sigma\beta_B s+p_\perp^2}-\frac{2p_ik_k}{k_\perp^2(\sigma\beta_B s+p_\perp^2)}\Big)\bigg]\\
&\quad\times\langle p|\tilde{\mathcal{F}}^k\Big(\beta_B+\frac{k_\perp^2}{\sigma s},p_\perp-k_\perp\Big)\mathcal{F}^l\Big(\beta_B+\frac{k_\perp^2}{\sigma s},p'_\perp-k_\perp\Big)|p\rangle\\
&\quad+\frac{2}{k_\perp^2}\bigg[\frac{(2k^lp'_j-k_jp'^l)\delta^k_i}{\sigma\beta_B s+(p'+k)_\perp^2}+\frac{(2p_ik^k-k_ip^k)\delta^l_j}{\sigma\beta_B s+(p+k)_\perp^2}\bigg]\langle p|\tilde{\mathcal{F}}^a_k(\beta_B,p_\perp)\mathcal{F}^a_l(\beta_B,p'_\perp)|p\rangle\\
&\quad-\frac{4}{k_\perp^2}\langle p|\bigg[\theta\Big(1-\beta_B-\frac{k_\perp^2}{\sigma s}\Big)\tilde{\mathcal{F}}_i\Big(\beta_B+\frac{k_\perp^2}{\sigma s},p_\perp-k_\perp\Big)\mathcal{F}_j\Big(\beta_B+\frac{k_\perp^2}{\sigma s},p'_\perp-k_\perp\Big)\\
&\quad-\frac{\sigma\beta_B s}{\sigma\beta_B s+k_\perp^2}\tilde{\mathcal{F}}^a_i(\beta_B,p_\perp)\mathcal{F}^a_j(\beta_B,p'_\perp)\bigg]|p\rangle\bigg\}
\end{aligned}
$$

where we performed Fourier transformation to the momentum space. Also, the forward matrix element $\langle p|\tilde{\mathcal{F}}_i(\beta_B,p_\perp)\mathcal{F}_j(\beta_B,p'_\perp)|p\rangle$ is proportional to $\delta^{(2)}(p_\perp-p'_\perp)$. Eliminating this factor and rewriting in terms of $\mathcal{R}_{ij}$ (see eq. (2.9)) we obtain ($\eta \equiv \ln\sigma$)

$$
\begin{aligned}
&\frac{d}{d\eta}\mathcal{R}_{ij}(\beta_B, p_\perp;\eta) \qquad (7.2)\\
&= -\alpha_s N_c \int d\!\!\!^-{}^2 k_\perp \Bigg\{ \Bigg[ \Bigg( \frac{(2p-k)_k}{\sigma\beta_B s + p_\perp^2} \frac{\sigma\beta_B s g_{\mu i} - 2(p-k)^\perp_\mu (p-k)_i}{\sigma\beta_B s + (p-k)_\perp^2} - 2\frac{(p-k)^\perp_\mu g_{ik} + p_i g_{\mu k}}{\sigma\beta_B s + p_\perp^2} \Bigg)\\
&\quad \times \Bigg( \frac{\sigma\beta_B s \delta^\mu_j - 2(p-k)^\mu_\perp (p-k)_j}{\sigma\beta_B s + (p-k)_\perp^2} \frac{(2p-k)_l}{\sigma\beta_B s + p_\perp^2} - 2\frac{(p-k)^\mu_\perp g_{jl} + \delta^\mu_l p_j}{\sigma\beta_B s + p_\perp^2} \Bigg)\\
&\quad +2g_{ik}\Bigg( \frac{(p-k)_j(2p-k)_l - 2p_j(p-k)_l}{(p-k)_\perp^2(\sigma\beta_B s + p_\perp^2)} + \frac{(p-k)_j(2p-k)_l}{(\sigma\beta_B s + (p-k)_\perp^2)(\sigma\beta_B s + p_\perp^2)} + \frac{2g_{jl}}{\sigma\beta_B s + p_\perp^2} \Bigg)\\
&\quad +2g_{lj}\Bigg( \frac{(p-k)_i(2p-k)_k - 2p_i(p-k)_k}{(p-k)_\perp^2(\sigma\beta_B s + p_\perp^2)} + \frac{(p-k)_i(2p-k)_k}{(\sigma\beta_B s + (p-k)_\perp^2)(\sigma\beta_B s + p_\perp^2)} + \frac{2g_{ik}}{\sigma\beta_B s + p_\perp^2} \Bigg)\Bigg]\\
&\quad \times\theta\Bigg(1-\beta_B - \frac{(p-k)_\perp^2}{\sigma s}\Bigg)\mathcal{R}^{kl}\Bigg(\beta_B + \frac{(p-k)_\perp^2}{\sigma s}, k_\perp\Bigg)\\
&\quad +2\frac{\delta^k_i(k_j p^l - 2k^l p_j) + \delta^l_j(k_i p^k - 2p_i k^k)}{k_\perp^2[\sigma\beta_B s + (p-k)_\perp^2]}\mathcal{R}_{kl}(\beta_B, p_\perp;\eta)\\
&\quad -4\Bigg[\frac{\theta\Big(1-\beta_B - \frac{(p-k)_\perp^2}{\sigma s}\Big)}{(p-k)_\perp^2}\mathcal{R}_{ij}\Bigg(\beta_B + \frac{(p-k)_\perp^2}{\sigma s}, k_\perp;\eta\Bigg) - \frac{\sigma\beta_B s}{k_\perp^2(\sigma\beta_B s + k_\perp^2)}\mathcal{R}_{ij}(\beta_B, p_\perp;\eta)\Bigg]\Bigg\}
\end{aligned}
$$

Let us demonstrate that eq. (7.2) reduces to BFKL equation in the low-$x$ limit. Indeed, in this limit $\mathcal{R}_{ij}$ is proportional to the WW distribution (1.1): $\mathcal{R}_{ij}(0,k_\perp) \sim \int d^2x e^{i(k,x)_\perp}\langle p|\mathrm{tr}\{\tilde{U}_i(x)U_j(0)\}|p\rangle$. In the leading-order BFKL approximation (cf. ref. [17–20])

$$
\begin{aligned}
&\langle p|\mathrm{tr}\{\tilde{U}_i(x)U_j(y)\}|p\rangle \qquad (7.3)\\
&= \frac{\alpha_s}{4\pi^2}\int \frac{d^2q_\perp}{q_\perp^2} q_i q_j e^{i(q,x)_\perp - i(q,y)_\perp} \int \frac{d^2q'_\perp}{q'^2_\perp}\Phi_T(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\left(\frac{s}{qq'}\right)^\omega G_\omega(q,q')
\end{aligned}
$$

Here $\Phi_T(q')$ is the target impact factor and $G_\omega(q,q')$ is the partial wave of the forward reggeized gluon scattering amplitude satisfying the equation

$$
\omega G_\omega(q,q') = \delta^{(2)}(q-q') + \int d^2p K_{\mathrm{BFKL}}(q,p)G_\omega(p,q') \qquad (7.4)
$$

with the forward BFKL kernel

$$
K_{\mathrm{BFKL}}(q,p) = \frac{\alpha_s N_c}{\pi^2}\left[\frac{1}{(q-p)_\perp^2} - \frac{1}{2}\delta(q_\perp - p_\perp)\int dp'_\perp \frac{q_\perp^2}{p'^2_\perp(q-p')_\perp^2}\right]
$$

Thus, in the BFKL approximation

$$
\mathcal{R}_{ij}(0,q_\perp;\ln\sigma) = q_i q_j R(q_\perp;\ln\sigma) = \frac{\alpha_s q_i q_j}{2\pi^2 q_\perp^2}\int \frac{d^2q'}{q'^2}\Phi_T(q')\int_{a-i\infty}^{a+i\infty}\frac{d\omega}{2\pi i}\left(\frac{\sigma s}{qq'}\right)^\omega G_\omega(p,q') \qquad (7.5)
$$

and the equation for $R$

$$\frac{d}{d\ln\sigma}R^\sigma(q_\perp;\ln\sigma) = \int d^2p_\perp\,\frac{p_\perp^2}{q_\perp^2}K_{\rm BFKL}(q,p)R^\sigma(p_\perp;\ln\sigma) \tag{7.6}$$

is obtained by differentiation of eq. (7.5) with respect to $\ln\sigma$ using eq. (7.4).

Now it is easy to see that our eq. (7.2) reduces to eq. (7.6) in the BFKL limit. As we discussed above, in this limit one may set $\beta_B = 0$ and neglect $\frac{k_\perp^2}{\sigma s}$ in the argument of $\mathcal{R}_{ij}$. Substituting $\mathcal{R}_{ij}(0,k_\perp) = k_ik_jR(k_\perp)$ into eq. (7.2) one obtains after some algebra

$$\frac{d}{d\ln\sigma}R(p_\perp;\ln\sigma) = 2\alpha_sN_c\int d\!\!{}^-{}^2k_\perp\Big[\frac{2k_\perp^2}{p_\perp^2(p-k)_\perp^2}R(k_\perp;\ln\sigma) - \frac{p_\perp^2}{k_\perp^2(p-k)_\perp^2}R(p_\perp;\ln\sigma)\Big]$$

which coincides with eq. (7.6). We have also checked that eq. (7.1) at $p_\perp \neq p'_\perp$ reduces to the non-forward BFKL equation in the low-$x$ limit.

Let us check now that the evolution of

$$\mathcal{D}(\beta_B,\ln\sigma) = -\frac{1}{2}\int d\!\!{}^-{}^2p_\perp\mathcal{R}_i^{\ i}(\beta_B,p_\perp;\ln\sigma) \tag{7.7}$$

reduces to DGLAP equation. As we discussed above, in the light-cone limit one can neglect $k_\perp$ in comparison to $p_\perp$. Indeed, the integral over $p_\perp$ converges at $p_\perp^2 \sim \sigma\beta_Bs$. On the other hand, extra $k_ik_j$ in the integral over $k_\perp$ leads to the operators of higher collinear twist, for example

$$\begin{aligned}\int d^2k_\perp\ k_ik_j\ R_n^{\ n}(\beta_B,k_\perp;\ln\sigma)\ &\sim\ \langle p|\partial_k\tilde{\mathcal{F}}_n^a(\beta_B,0_\perp)\partial_j\mathcal{F}^{an}(\beta_B,0_\perp)|p\rangle^{\eta=\ln\sigma}\\ \sim\ m^2g_{ij}\langle p|\tilde{\mathcal{F}}_n^a(\beta_B,0_\perp)\mathcal{F}^{an}(\beta_B,0_\perp)|p\rangle^{\ln\sigma}\ &\sim\ m^2\mathcal{D}(\beta_B,\ln\sigma)\end{aligned} \tag{7.8}$$

(where $m$ is the mass of the target) so $\frac{k_\perp^2}{p_\perp^2} \sim \frac{k_\perp^2}{\sigma\beta_Bs} \sim \frac{m^2}{\sigma s} \ll 1$.

Neglecting $k_\perp$ in comparison to $p_\perp$ and integrating over angles one obtains

$$\begin{aligned}&\frac{d}{d\ln\sigma}\int d^2p_\perp\ \mathcal{R}_i^{\ i}(\beta_B,p_\perp;\ln\sigma)\\ &= \frac{\alpha_sN_c}{\pi^2}\int d^2p_\perp\Big[\frac{1}{p_\perp^2} - \frac{2}{\sigma\beta_Bs+p_\perp^2} + \frac{3p_\perp^2}{(\sigma\beta_Bs+p_\perp^2)^2} - \frac{2p_\perp^4}{(\sigma\beta_Bs+p_\perp^2)^3} + \frac{p_\perp^6}{(\sigma\beta_Bs+p_\perp^2)^4}\Big]\\ &\quad\times\int d^2k_\perp\mathcal{R}_i^{\ i}\Big(\beta_B+\frac{p_\perp^2}{\sigma s},k_\perp;\ln\sigma\Big) - \int d^2k_\perp\frac{\sigma\beta_Bs}{k_\perp^2(\sigma\beta_Bs+k_\perp^2)}\int d^2p_\perp\mathcal{R}_i^{\ i}(\beta_B,p_\perp;\ln\sigma)\end{aligned}$$

which coincides with DGLAP equation (3.33).

It would be interesting to compare eq. (7.2) to CCFM equation [41–43] which also addresses the question of interplay of BFKL and DGLAP logarithms.

## 8 Rapidity evolution of fragmentation functions

In this section we will construct the evolution equation for fragmentation function (1.7). We start from eq. (5.2) which enables us to analytically continue to negative $\beta_B = -\beta_F$. In the operator form, the equation (5.2) has imaginary parts at negative $\beta_B = -\beta_F$ corresponding to poles of propagators $(\sigma\beta_F s - p_\perp^2 \pm i\epsilon)^{-1}$ but we will demonstrate now that for the evolution of a "fragmentation matrix element" (2.10)[9]

$$\langle \tilde{\mathcal{F}}_i^a(-\beta_F, x_\perp)\mathcal{F}_j^a(-\beta_F, y_\perp)\rangle_{\text{frag}} \equiv \sum_X \langle 0|\tilde{\mathcal{F}}_i^a(-\beta_F, x_\perp)|p+X\rangle\langle p+X|\mathcal{F}_j^a(-\beta_F, y_\perp)|0\rangle \quad (8.1)$$

we have the kinematical restriction $\sigma(\beta_F - 1)s > p_\perp^2$ in all the integrals in the production part of the kernel (5.2). As to virtual part of the kernel, we will see that the imaginary parts there assemble to yield the principle-value prescription for integrals over $p_\perp^2$. The "fragmentation matrix element" (2.10) of eq. (5.4) has the form

$$\begin{aligned}
&\frac{d}{d\ln\sigma}\langle \tilde{\mathcal{F}}_i^a(-\beta_F, x_\perp)\mathcal{F}_j^a(-\beta_F, y_\perp)\rangle_{\text{frag}} \qquad (8.2)\\
&= -\alpha_s\langle \mathrm{Tr}\Big\{\int {\not d}^2k_\perp \theta\Big(\beta_F - 1 - \frac{k_\perp^2}{\sigma s}\Big)\Big[(x_\perp|\Big(\tilde{U}\frac{-1}{\sigma\beta_F s - p_\perp^2}(\tilde{U}^\dagger k_k + p_k\tilde{U}^\dagger)\frac{\sigma\beta_F s g_{\mu i} + 2k_\mu^\perp k_i}{\sigma\beta_F s - k_\perp^2}\\
&\quad +2k_\mu^\perp g_{ik}\tilde{U}\frac{1}{\sigma\beta_F s - p_\perp^2}\tilde{U}^\dagger + 2g_{\mu k}\tilde{U}\frac{p_i}{\sigma\beta_F s - p_\perp^2}\tilde{U}^\dagger\Big)\tilde{\mathcal{F}}^k\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad \times(k_\perp|\mathcal{F}^l\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}\Big)\Big(\frac{\sigma\beta_F s\delta_j^\mu + 2k_\perp^\mu k_j}{\sigma\beta_F s - k_\perp^2}(k_l U + U p_l)\frac{-1}{\sigma\beta_F s + p_\perp^2}U^\dagger\\
&\quad +2k_\perp^\mu g_{jl}U\frac{1}{\sigma\beta_F s - p_\perp^2}U^\dagger + 2\delta_l^\mu U\frac{p_j}{\sigma\beta_F s - p_\perp^2}U^\dagger\Big)|y_\perp) + 2(x_\perp|\tilde{\mathcal{F}}_i\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}\Big)|k_\perp)\\
&\quad \times(k_\perp|\mathcal{F}^l\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}\Big)\Big(-\frac{k_j}{k_\perp^2}\frac{\sigma\beta_F s - 2k_\perp^2}{\sigma\beta_F s - k_\perp^2}(k_l U + U p_l)\frac{1}{\sigma\beta_F s - p_\perp^2}U^\dagger\\
&\quad -2U\frac{g_{jl}}{\sigma\beta_F s - p_\perp^2}U^\dagger + 2\frac{k_l}{k_\perp^2}U\frac{p_j}{\sigma\beta_F s - p_\perp^2}U^\dagger\Big)|y_\perp)\\
&\quad +2(x_\perp|\Big(-\tilde{U}\frac{1}{\sigma\beta_F s - p_\perp^2}(\tilde{U}^\dagger k_k + p_k\tilde{U}^\dagger)\frac{k_i}{k_\perp^2}\frac{\sigma\beta_F s - 2k_\perp^2}{\sigma\beta_F s - k_\perp^2} - 2\tilde{U}\frac{g_{ik}}{\sigma\beta_F s - p_\perp^2}\tilde{U}^\dagger\\
&\quad +2\tilde{U}\frac{p_i}{\sigma\beta_F s - p_\perp^2}\tilde{U}^\dagger\frac{k_k}{k_\perp^2}\Big)\tilde{\mathcal{F}}^k\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}\Big)|k_\perp)(k_\perp|\mathcal{F}_j\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}\Big)|y_\perp)\Big]\\
&\quad +2\tilde{\mathcal{F}}_i(-\beta_F, x_\perp)(y_\perp|\frac{p^m}{p_\perp^2}\mathcal{F}_k(-\beta_F)(i\overleftarrow{\partial}_l + U_l)(2\delta_m^k\delta_j^l - g_{jm}g^{kl})U\frac{1}{\sigma\beta_F s - p_\perp^2 + i\epsilon}U^\dagger|y_\perp)\\
&\quad +2(x_\perp|\tilde{U}\frac{-1}{\sigma\beta_F s - p_\perp^2 - i\epsilon}\tilde{U}^\dagger(2\delta_i^k\delta_m^l - g_{im}g^{kl})(i\partial_k - \tilde{U}_k)\tilde{\mathcal{F}}_l(-\beta_F)\frac{p^m}{p_\perp^2}|x_\perp)\mathcal{F}_j(-\beta_F, y_\perp)\\
&\quad -4\int\frac{{\not d}^2k_\perp}{k_\perp^2}\Big[\theta\Big(\beta_F - 1 - \frac{k_\perp^2}{\sigma s}\Big)\tilde{\mathcal{F}}_i\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}, x_\perp\Big)\mathcal{F}_j\Big(-\beta_F + \frac{k_\perp^2}{\sigma s}, y_\perp\Big)e^{i(k,x-y)_\perp}\\
&\quad -\frac{1}{2}\Big[\frac{\sigma\beta_F s}{\sigma\beta_F s - k_\perp^2 - i\epsilon} + \frac{\sigma\beta_F s}{\sigma\beta_F s - k_\perp^2 + i\epsilon}\Big]\tilde{\mathcal{F}}_i(-\beta_F, x_\perp)\mathcal{F}_j(-\beta_F, y_\perp)\Big]\Big\}\rangle_{\text{frag}} + O(\alpha_s^2)
\end{aligned}$$

where we have restored $\pm i\epsilon$ in the virtual part in accordance with Feynman rules.

[9]Again, strictly speaking we should consider $\sum_X\langle 0|\tilde{\mathcal{F}}_i^a(-\beta_F, x_\perp)|p+X\rangle\langle (p+\xi p_2)+X|\mathcal{F}_j^a(-\beta_F, y_\perp)|0\rangle$, see eq. (2.10).

Let us prove that all non-linear terms in eq. (8.2) can be neglected with our accuracy. (Naïvely, they were important at small $\beta_B$ but small $\beta_F$ are not allowed due to kinematical restrictions). First, consider the "light-cone" case when the transverse momenta of fast fields $l_\perp^2$ are smaller than the characteristic transverse momenta in the gluon loop of slow fields $p_\perp^2 \sim k_\perp^2$. As we discussed above, in this case with the leading-twist accuracy we can commute all $U$'s with $p_\perp$ operators until they form $UU^\dagger = 1$ and disappear. In this limit the (8.2) turns to

$$
\begin{aligned}
&\frac{d}{d\ln\sigma}\langle\tilde{\mathcal{F}}_i^a(-\beta_F,x_\perp)\mathcal{F}_j^a(-\beta_F,y_\perp)\rangle_{\text{frag}} = -4\alpha_s\mathrm{Tr}\Big\{(x_\perp|\theta\left(\beta_F-1-\frac{p_\perp^2}{\sigma s}\right)\\
&\times\left[\frac{\delta_i^k p_\perp^\mu}{p_\perp^2}+\frac{g^{\mu k}p_i+p_\perp^\mu\delta_i^k-\delta_i^\mu p^k}{\sigma\beta_F s-p_\perp^2-i\epsilon}-\frac{\delta_i^\mu p_\perp^2 p^k+2p_ip^kp_\perp^\mu}{(\sigma\beta_F s-p_\perp^2-i\epsilon)^2}\right]\left[\frac{p_\mu^\perp\delta_j^l}{p_\perp^2}+\frac{\delta_\mu^l p_j+\delta_j^l p_\mu^\perp-g_{\mu j}p^l}{\sigma\beta_F s-p_\perp^2+i\epsilon}\right.\\
&\left.-\frac{g_{\mu j}p_\perp^2p^l+2p_\mu^\perp p_jp^l}{(\sigma\beta_F s-p_\perp^2+i\epsilon)^2}\right]|y_\perp)\langle\tilde{\mathcal{F}}_k\left(\frac{p_\perp^2}{\sigma s}-\beta_F,x_\perp\right)\mathcal{F}_l\left(\frac{p_\perp^2}{\sigma s}-\beta_F,y_\perp\right)\rangle_{\text{frag}}\\
&+\frac{1}{2}\left[(x_\perp|\frac{\sigma\beta_F s}{p_\perp^2(\sigma\beta_F s-p_\perp^2-i\epsilon)}|x_\perp)+(y_\perp|\frac{\sigma\beta_F s}{p_\perp^2(\sigma\beta_F s-p_\perp^2+i\epsilon)}|y_\perp)\right]\\
&\times\langle\tilde{\mathcal{F}}_i(-\beta_F,x_\perp)\mathcal{F}_j(-\beta_F,y_\perp)\rangle_{\text{frag}}\Big\}
\end{aligned}
\tag{8.3}
$$

Now consider the shock-wave case when $l_\perp^2 \sim p_\perp^2$. There are two "subcases": when $\beta_F\sigma_* \geq 1$ and when $\beta_F\sigma_* \ll 1$ (where $\sigma_* \sim \frac{\sigma' s}{l_\perp^2}$). In the former case we have $\beta_F\frac{\sigma' s}{l_\perp^2} \geq 1$ so $\sigma\beta_F s \gg p_\perp^2$ and only two last lines in eq. (8.2) survive. Moreover, in this case eq. (8.3) also reduces to the last two lines so eq. (8.2) is equivalent to eq. (8.3) in this case as well.

If $\beta_F\sigma_* \ll 1$, as we discussed above, one can replace $\mathcal{F}_j(-\beta_F)$ (and $\mathcal{F}_j\big(-\beta_F+\frac{p_\perp^2}{\alpha s}\big)$) by $U_j$. We will prove now that after such replacement the r.h.s. of eq. (8.2) vanishes, and so does the r.h.s. of eq. (8.3), and therefore eqs. (8.2) and (8.3) are equivalent in the case of large $\beta_F\sigma_*$ also.

Let us now prove that if we replace all $\mathcal{F}_j(-\beta_F)$ and $\mathcal{F}_j\big(-\beta_F+\frac{p_\perp^2}{\alpha s}\big)$ by $U_j$ the r.h.s. of eq. (8.2) vanishes. Indeed, a typical term in Feynman part of the amplitude vanishes:

$$
\langle p+X|U_j(z)U_{z'}U_{z''}^\dagger|0\rangle = 0 \tag{8.4}
$$

To prove this, let us consider the shift of $U$ operator on $\frac{2}{s}a_*p_1$. Since the shift in the $p_1$ direction does not change the infinitely long $U$ operator, we get

$$
\begin{aligned}
\langle p+X|U_j(z)U_{z'}U_{z''}^\dagger|0\rangle &= \langle p+X|e^{i\frac{2}{s}\hat{P}_\bullet a_*}U_j(z)U_{z'}U_{z''}^\dagger e^{-i\frac{2}{s}\hat{P}_\bullet a_*}|0\rangle\\
&= e^{i(\beta_p+\beta_X)a_*}\langle p+X|U_j(z)U_{z'}U_{z''}^\dagger|0\rangle
\end{aligned}
$$

which can be true only if eq. (8.4) vanishes. It is clear that for the same reason all terms in the r.h.s. of eq. (8.2) (and r.h.s. of eq. (8.3) as well) vanish. Summarizing, in all regimes the eq. (8.2) can be reduced to the light-cone version (8.3).

One can rewrite eq. (8.3) in the form:

$$\frac{d}{d\ln\sigma}\langle\tilde{\mathcal{F}}^a_i(-\beta_F,x_\perp)\mathcal{F}^a_j(-\beta_F,y_\perp)\rangle^{\eta\equiv\ln\sigma}_{\rm frag} \tag{8.5}$$
$$= 4\alpha_sN_c\int d\!\!\!^-{}^2p_\perp\Big\{\theta\Big(\beta_F-1-\frac{p_\perp^2}{\sigma s}\Big)e^{i(p,x-y)_\perp}\langle\tilde{\mathcal{F}}^a_k\Big(\frac{p_\perp^2}{\sigma s}-\beta_F,x_\perp\Big)\mathcal{F}^a_l\Big(\frac{p_\perp^2}{\sigma s}-\beta_F,y_\perp\Big)\rangle^\eta_{\rm frag}$$
$$\times\Big[\frac{\delta^k_i\delta^l_j}{p_\perp^2}+\frac{2\delta^k_i\delta^l_j}{\sigma\beta_Fs-p_\perp^2}+\frac{p_\perp^2\delta^k_i\delta^l_j+\delta^k_jp_ip^l+\delta^l_ip_jp^k-\delta^l_jp_ip^k-\delta^k_ip_jp^l-g^{kl}p_ip_j-g_{ij}p^kp^l}{(\sigma\beta_Fs-p_\perp^2)^2}$$
$$-p_\perp^2\frac{2g_{ij}p^kp^l+\delta^k_ip_jp^l+\delta^l_jp_ip^k-\delta^k_jp_ip^l-\delta^l_ip_jp^k}{(\sigma\beta_Fs-p_\perp^2)^3}-\frac{p_\perp^4g_{ij}p^kp^l}{(\sigma\beta_Fs-p_\perp^2)^4}\Big]$$
$$-\frac{\theta(\sigma\beta_Fs-p_\perp^2)}{p_\perp^2}\langle\tilde{\mathcal{F}}^a_i(-\beta_F,x_\perp)\mathcal{F}^a_j(-\beta_F,y_\perp)\rangle^\eta_{\rm frag}\Big\}$$

where we used the formula

$$\int\frac{d\!\!\!^-{}^2p_\perp}{p_\perp^2}\frac{1}{2}\Big[\frac{\sigma\beta_Fs}{\sigma\beta_Fs-p_\perp^2+i\epsilon}+\frac{\sigma\beta_Fs}{\sigma\beta_Fs-p_\perp^2-i\epsilon}\Big]=\int\frac{d\!\!\!^-{}^2p_\perp}{p_\perp^2}\theta(\sigma\beta_Fs-p_\perp^2) \tag{8.6}$$

The eq. (8.5) is our final evolution equation for fragmentation functions valid for all $(x-y)^2_\perp$ (and all $\beta_F$).

If polarizations of fragmentation hadron are not registered we can use the parametrization (2.10)

$$\langle\tilde{\mathcal{F}}^a_i(-\beta_F,z_\perp)\mathcal{F}^a_j(-\beta_F,0_\perp)\rangle^\eta_{\rm frag}$$
$$=2\pi^2\delta(\xi)\beta_Fg^2\Big[-g_{ij}\mathcal{D}_{\rm f}(\beta_F,z_\perp,\eta)-\frac{4}{m^2}(2z_iz_j+g_{ij}z_\perp^2)\mathcal{H}''_{\rm f}(\beta_F,z_\perp,\eta)\Big] \tag{8.7}$$

where $\mathcal{H}_{\rm f}(\beta_F,z_\perp,\eta)\equiv\int d\!\!\!^-{}^2k_\perp\ e^{i(k,z)_\perp}\mathcal{H}_{\rm f}(\beta_F,k_\perp,\eta)$ and $\mathcal{H}''_{\rm f}(\beta_F,z_\perp,\eta)\equiv\big(\frac{\partial}{\partial z^2}\big)^2\mathcal{H}_{\rm f}(\beta_F,z_\perp,\eta)$, cf. eq. (3.26). After integration over angles similar to eq. (3.27) one obtains

$$\frac{d}{d\eta}\Big[g_{ij}\alpha_s\mathcal{D}_{\rm f}(\beta_F,z_\perp,\eta)+\frac{4}{m^2}(2z_iz_j+g_{ij}z_\perp^2)\alpha_s\mathcal{H}''_{\rm f}(\beta_F,z_\perp,\eta)\Big] \tag{8.8}$$
$$=\frac{\alpha_sN_c}{\pi}\int_0^{\beta_F-1}d\beta\ \Big\{g_{ij}J_0\big(|z_\perp|\sqrt{\sigma\beta}\big)\alpha_s\mathcal{D}_{\rm f}(\beta_F-\beta,z_\perp,\eta)$$
$$\times\Big[\frac{\beta_F-\beta}{\beta\beta_F}+\frac{2}{\beta_F}+\frac{3\beta}{\beta_F(\beta_F-\beta)}+\frac{2\beta^2}{\beta_F(\beta_F-\beta)^2}+\frac{\beta^3}{\beta_F(\beta_F-\beta)^3}\Big]$$
$$+J_2(|z_\perp|\sqrt{\sigma\beta})\Big(2\frac{z_iz_j}{z_\perp^2}+g_{ij}\Big)\alpha_s\mathcal{D}_{\rm f}(\beta_F-\beta,z_\perp,\eta)\frac{\beta}{\beta_F(\beta_F-\beta)}$$
$$+\frac{4}{m^2}J_0\big(|z_\perp|\sqrt{\sigma\beta}\big)(2z_iz_j+g_{ij}z_\perp^2)\alpha_s\mathcal{H}''_{\rm f}(\beta_F-\beta,z_\perp,\eta)\Big[\frac{\beta_F-\beta}{\beta_F\beta}+\frac{2}{\beta_F}+\frac{\beta}{\beta_F(\beta_F-\beta)}\Big]$$
$$+\frac{4g_{ij}}{m^2}z_\perp^2J_2\big(|z_\perp|\sqrt{\sigma\beta}\big)\alpha_s\mathcal{H}''_{\rm f}(\beta_F-\beta,z_\perp,\eta)$$
$$\times\Big[\frac{\beta}{\beta_F(\beta_F-\beta)}+\frac{2\beta^2}{\beta_F(\beta_F-\beta)^2}+\frac{\beta^3}{\beta_F(\beta_F-\beta)^3}\Big]\Big\}$$
$$-\frac{\alpha_sN_c}{\pi}\int_0^{\beta_F}\frac{d\beta}{\beta}\Big[g_{ij}\alpha_s\mathcal{D}_{\rm f}(\beta_F,z_\perp,\eta)+\frac{4}{m^2}(2z_iz_j+g_{ij}z_\perp^2)\alpha_s\mathcal{H}''_{\rm f}(\beta_F,z_\perp,\eta)\Big]$$

where $\beta\equiv\frac{p_\perp^2}{\sigma s}$ (and $\sigma\equiv e^\eta$ as usual).

This evolution equation can be rewritten as a system (cf. eq. (3.29) for DIS)

$$
\begin{aligned}
&\frac{d}{d\eta}\alpha_s\mathcal{D}_{\rm f}\left(\frac{1}{z_F},z_\perp,\eta\right) \qquad (8.9)\\
&=\frac{\alpha_sN_c}{\pi}\int_{z_F}^1\frac{dz'}{z'^2}\left\{J_0\left(|z_\perp|\sqrt{\frac{\sigma s}{z_F}(1-z')}\right)\left[\left(\frac{1}{1-z'}\right)_+ +\frac{1}{z'}-2+z'(1-z')\right]\alpha_s\mathcal{D}_{\rm f}\left(\frac{z'}{z_F},z_\perp,\eta\right)\right.\\
&\quad\left.+\frac{4}{m^2}\frac{1-z'}{z'}z_\perp^2J_2\left(|z_\perp|\sqrt{\frac{\sigma s}{z_F}(1-z')}\right)\alpha_s\mathcal{H}''_{\rm f}\left(\frac{z'}{z_F},z_\perp,\eta\right)\right\},\\
&\frac{d}{d\eta}\alpha_s\mathcal{H}''_{\rm f}\left(\frac{1}{z_F},z_\perp,\eta\right)\\
&=\frac{\alpha_sN_c}{\pi}\int_{z_F}^1\frac{dz'}{z'^2}\left\{J_0\left(|z_\perp|\sqrt{\frac{\sigma s}{z_F}(1-z')}\right)\left[\left(\frac{1}{1-z'}\right)_+ -1\right]\alpha_s\mathcal{H}''_{\rm f}\left(\frac{z'}{z_F},z_\perp,\eta\right)\right.\\
&\quad\left.+\frac{m^2}{4z_\perp^2}(1-z')z'J_2\left(|z_\perp|\sqrt{\frac{\sigma s}{z_F}(1-z')}\right)\alpha_s\mathcal{D}_{\rm f}\left(\frac{z'}{z_F},z_\perp,\eta\right)\right\}
\end{aligned}
$$

where $z'=1-\frac{p_\perp^2}{\sigma\beta_Fs}=1-\beta z_F$. Here we introduced the standard notation $z_F\equiv\frac{1}{\beta_F}$ for the fraction of the "initial gluon momentum" carried by the hadron. By construction, this equation describes the evolution of fragmentation TMD at any $z_F$ and any $k_\perp\sim|z_\perp|^{-1}$.

Let us demonstrate that eq. (8.9) agrees with the DGLAP equation for fragmentation functions in the light-cone limit $x_\perp\to y_\perp$. In this limit

$$
\begin{aligned}
\frac{d}{d\ln\sigma}\alpha_s\mathcal{D}^{\rm f}\left(\frac{1}{z_F},0_\perp,\ln\sigma s\right)&=\frac{\alpha_s}{\pi}N_c\left\{\int_{z_F}^1\frac{dz'}{z'^2}\left[\frac{1}{z'(1-z')}+z'(1-z')-2\right]\right.\\
&\left.\times\alpha_s\mathcal{D}^{\rm f}\left(\frac{z'}{z_F},0_\perp,\ln\sigma s\right)-\alpha_s\mathcal{D}^{\rm f}\left(\frac{1}{z_F},0_\perp,\ln\sigma s\right)\int_0^1\frac{dz'}{1-z'}\right\}
\end{aligned}
\tag{8.10}
$$

As explained in eq. (3.30), with leading-log accuracy we can trade the cutoff in $\alpha$ for cutoff in $\mu^2$. In terms of the standard definition of fragmentation functions [3]

$$
d_g^{\rm f}(z_F,\ln\mu^2)=-\frac{z_F^2}{16\pi^2\alpha_sN_c(p\cdot n)}\int du\ e^{i\frac{u}{z_F}(pn)}\sum_X\langle 0|\tilde{\mathcal{F}}_\xi^a(un)|p+X\rangle\langle p+X|\mathcal{F}^{a\xi}(0)|0\rangle^\mu
\tag{8.11}
$$

we have in the leading log approximation

$$
d_g^{\rm f}(z_F,\ln\mu^2)=\frac{z_F}{2N_c}\mathcal{D}^{\rm f}\left(\frac{1}{z_F},z_\perp=0,\ln\sigma s\right)+O(\alpha_s)
\tag{8.12}
$$

so we can rewrite eq. (8.10) in the form

$$
\begin{aligned}
&\frac{d}{d\ln\mu^2}\alpha_s(\mu)d_g^{\rm f}(z_F,\ln\mu^2)\\
&=\frac{\alpha_s(\mu)}{\pi}N_c\int_{z_F}^1\frac{dz'}{z'}\left(\left[\frac{1}{1-z'}\right]_+ +\frac{1}{z'}+z'(1-z')-2\right)\alpha_s(\mu)d_g^{\rm f}\left(\frac{z_F}{z'},\ln\mu^2\right)
\end{aligned}
\tag{8.13}
$$

easily recognizable as the DGLAP equation for fragmentation functions [31–33]. (Here again the term proportional to $\beta$-function is absent since $\tilde{F}_i^aF^{ai}$ is defined with an extra $\alpha_s$.)

Finally, let us describe what happens if $z_F \ll 1$ and we evolve from $\sigma \sim 1$ to $\sigma \sim \frac{z_F}{z_\perp^2 s}$. With double-log accuracy we have an equation

$$\frac{d}{d\eta}\mathcal{D}^{\rm f}\left(\frac{1}{z_F}, z_\perp, \eta\right) = -\frac{\alpha_s N_c}{\pi^2}\mathcal{D}^{\rm f}\left(\frac{1}{z_F}, z_\perp, \eta\right)\int \frac{d^2 p_\perp}{p_\perp^2}\left[1 - e^{i(p,z)_\perp}\right] \tag{8.14}$$

(cf. eq. (6.8)) with the solution of the Sudakov type

$$\mathcal{D}^{\rm f}\left(\frac{1}{z_F}, z_\perp, \ln\sigma\right) \sim \exp\left\{-\frac{\alpha_s N_c}{2\pi}\ln^2\frac{\sigma}{z_F}s z_\perp^2\right\}\mathcal{D}^{\rm f}\left(\frac{1}{z_F}, z_\perp, \ln\frac{z_F}{z_\perp^2 s}\right) \tag{8.15}$$

The evolution with the single-log accuracy should be determined from the full system (8.9).

## 9 Conclusions

We have described the rapidity evolution of gluon TMD (1.6) with Wilson lines going to $+\infty$ in the whole range of Bjorken $x_B$ and the whole range of transverse momentum $k_\perp$. It should be emphasized that with our definition of rapidity cutoff (2.1) the leading-order matrix elements of TMD operators are UV-finite so the rapidity evolution is the only evolution and it describes all the dynamics of gluon TMDs (1.6) in the leading-log approximation.

The evolution equation for the gluon TMD (1.6) with rapidity cutoff (2.1) is given by (5.5) and, in general, is non-linear. Nevertheless, for some specific cases the equation (5.5) linearizes. For example, let us consider the case when $x_B \sim 1$. If in addition $k_\perp^2 \sim s$, the non-linearity can be neglected for the whole range of evolution $1 \gg \sigma \gg \frac{m_N^2}{s}$ and we get the DGLAP-type system of equations (3.29). If $k_\perp$ is small ($\sim$ few GeV) the evolution is linear and leads to usual Sudakov factors (6.10). If we consider now the intermediate case $x_B \sim 1$ and $s \gg k_\perp^2 \gg m_N^2$ the evolution at $1 \gg \sigma \gg \frac{k_\perp^2}{s}$ will be Sudakov-type (see eq. (6.6)) but the evolution at $\frac{k_\perp^2}{s} \gg \sigma \gg \frac{m_N^2}{s}$ will be described by the full master equation (5.5).

For low-x region $k_\perp \sim$ few GeV and $x_B \sim \frac{k_\perp^2}{s}$ we get the non-linear evolution described by the BK-type equation (6.3). If we now keep $k_\perp^2 \sim$ few GeV and take the intermediate $1 \gg x_B \equiv \beta_B \gg \frac{k_\perp^2}{s}$ we get a mixture of linear and non-linear evolutions. If one evolves $\sigma$ ($\leftrightarrow$ rapidity) from 1 to $\frac{k_\perp^2}{s}$ first there will be Sudakov-type double-log evolution (6.8) from $\sigma = 1$ to $\sigma = \frac{k_\perp^2}{\beta_B s}$, then the transitional region at $\sigma \sim \frac{k_\perp^2}{\beta_B s}$, and after that the non-linear evolution (6.3) at $\frac{k_\perp^2}{\beta_B s} \gg \sigma \gg \frac{k_\perp^2}{s}$. The transition between the linear evolution (6.8) and the non-linear one (6.3) should be described by the full equation (5.5).

Another interesting case is $x_B \sim \frac{m_N^2}{s}$ and $s \gg k_\perp^2 \gg m_N^2$. In this case, if we evolve $\sigma$ from 1 to $\frac{m_N^2}{s}$, first we have the BK evolution (6.3) up to $\sigma \sim \frac{k_\perp^2}{s}$ and then for the evolution between $\sigma \sim \frac{k_\perp^2}{s}$ and $\sigma \sim \frac{m_N^2}{s}$ we need the eq. (5.5) in full.

In conclusion, let us again emphasize that the evolution of the fragmentation TMDs (2.10) is always linear and the corresponding equation (8.8) describes both the DGLAP region $k_\perp^2 \sim s$ and Sudakov region $k_\perp^2 \sim$ few GeV$^2$.

As an outlook, it would be very interesting to obtain the NLO correction to the evolution equation (5.5). The NLO corrections to the BFKL [44, 45] and BK [17–20, 46–49] equation are available but they suffer from the well-known problem that they lead to negative cross sections. This difficulty can be overcome by the "collinear resummation" of double-logarithmic contributions for the BFKL [50–53] and BK [54–56] equations and we hope that our eq. (5.5) and especially its future NLO version will help to solve the problem of negative cross sections of NLO amplitudes at high energies.


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## A Light-cone expansion of propagators

In this section we consider the case when the transverse momenta of background fast fields $l_\perp$ are much smaller than the characteristic transverse momenta $p_\perp$ of "quantum" slow gluons. As we discussed in section 2, in this case fast fields do not necessarily shrink to a shock wave and one should use the light-cone expansion of propagators instead. The parameter of expansion is the twist of the operator and we will expand up to operators of leading collinear twist two. Such operators are built of two gluon operators $\sim F_{\bullet i}F_{\bullet j}$ or quark ones $\bar\psi\, \not{p}_1\psi$ and gauge links. To get coefficients in front of these operators it is sufficient to consider the external gluon field of the type $A_\bullet(z_*, z_\perp)$ and quark fields $\not{p}_1\psi(x_*, x_\perp)$ with all other components being zero.[10]

For simplicity, let us again start with the expansion of a scalar propagator.

### A.1 Scalar propagator

#### A.1.1 Feynman propagator for a scalar particle in the background gluon field

For simplicity we will first perform the calculation for "scalar propagator" $(x|\frac{1}{P^2+i\epsilon}|y)$. As we mentioned above, we assume that the only nonzero component of the external field is $A_\bullet$ and it does not depend on $z_\bullet$ so the operator $\alpha = i\frac{\partial}{\partial z_\bullet}$ commutes with all background fields. The propagator in the external field $A_\bullet(z_*, z_\perp)$ has the form

$$(x|\frac{1}{P^2+i\epsilon}|y) = \Big[-i\theta(x_*-y_*)\int_0^\infty \frac{d\!\!{}^-\alpha}{2\alpha} + i\theta(y_*-x_*)\int_{-\infty}^0 \frac{d\!\!{}^-\alpha}{2\alpha}\Big] \times e^{-i\alpha(x-y)_\bullet}(x_\perp|\mathrm{Pexp}\Big\{-i\int_{y_*}^{x_*} dz_*\Big[\frac{p_\perp^2}{\alpha s} - \frac{2g}{s}A_\bullet(z_*)\Big]\Big\}|y_\perp) \tag{A.1}$$

[10]The $z_\bullet$ dependence of the external fields can be omitted since due to the rapidity ordering $\alpha$'s of the fast fields are much less than $\alpha$'s of the slow ones.

The Pexp in the r.h.s. of eq. (A.1) can be transformed to

$$
\begin{aligned}
&(x_\perp|\mathrm{Pexp}\Big\{-i\int_{y_*}^{x_*}dz_*\Big[\frac{p_\perp^2}{\alpha s}-\frac{2g}{s}A_\bullet(z_*)\Big]\Big\}|y_\perp)\\
&=(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x_*-y_*)}\mathrm{Pexp}\Big\{\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}A_\bullet(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big\}|y_\perp)
\end{aligned}
\tag{A.2}
$$

Since the longitudinal distances $z_*$ inside the shock wave are small we can expand

$$
\begin{aligned}
&e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}A_\bullet e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}=A_\bullet-\frac{z_*-y_*}{\alpha s}\{p^i,F_{\bullet i}\}-\frac{(z_*-y_*)^2}{2\alpha^2s^2}\{p^j,\{p^i,D_jF_{\bullet i}\}\}+\dots\\
&\quad=A_\bullet-\frac{z_*-y_*}{\alpha s}(2p^iF_{\bullet i}-iD^iF_{\bullet i})-2\frac{(z_*-y_*)^2}{\alpha^2s^2}(p^ip^j-ip^jD^i)D_jF_{\bullet i}+\dots
\end{aligned}
\tag{A.3}
$$

This is effectively expansion around the light ray $y_\perp+\frac{2}{s}y_*p_1$ with the parameter of the expansion $\sim\frac{|l_\perp|}{|p_\perp|}\ll 1$. As we mentioned, we will expand up to the operator(s) with twist two.

We obtain

$$
\mathcal{O}(x_*,y_*;p_\perp)\equiv\mathrm{Pexp}\Big\{\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}A_\bullet(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big\}
\tag{A.4}
$$

$$
\begin{aligned}
&=1+\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ \Big[A_\bullet-\frac{(z-y)_*}{\alpha s}(2p^iF_{\bullet i}-iD^iF_{\bullet i})-2\frac{(z_*-y_*)^2}{\alpha^2s^2}(p^ip^j-ip^jD^i)D_jF_{\bullet i}\Big]\\
&\quad-\frac{4g^2}{s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\Big[A_\bullet-\frac{2(z-y)_*}{\alpha s}p^iF_{\bullet i}\Big]\Big[A_\bullet-\frac{2(z'-y)_*}{\alpha s}p^jF_{\bullet j}\Big]+\dots
\end{aligned}
$$

It is clear that the terms $\sim A_\bullet$ will combine to form gauge links so the r.h.s. of the above equation will turn to

$$
\begin{aligned}
\mathcal{O}(x_*,y_*;p_\perp)\ &=[x_*,y_*]-\frac{2ig}{\alpha s^2}\int_{y_*}^{x_*}dz_*\ (z-y)_*\Big(2p^j[x_*,z_*]F_{\bullet j}(z_*)-i[x_*,z_*]D^jF_{\bullet j}(z_*)\\
&+2\frac{(z-y)_*}{\alpha s}(p^jp^k[x_*,z_*]-ip^k[x_*,z_*]D^j)D_kF_{\bullet j}\Big)[z_*,y_*]\\
&+\frac{8g^2}{\alpha s^3}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\ (z'-y)_*\Big(i[x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_\bullet{}^j(z'_*)\\
&-2p^jp^k\frac{(z-y)_*}{\alpha s}[x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_{\bullet k}(z'_*)\Big)[z'_*,y_*]+\dots
\end{aligned}
\tag{A.5}
$$

where dots stand for the higher twists.

Thus, the final expansion of the propagator (A.1) near the light cone $y_\perp+\frac{2}{s}y_*p_1$ takes the form

$$
\begin{aligned}
(x|\frac{1}{P^2+i\epsilon}|y)&=\Big[-i\theta(x_*-y_*)\int_0^\infty\frac{đ\alpha}{2\alpha}+i\theta(y_*-x_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]\\
&\qquad\times e^{-i\alpha(x-y)_\bullet}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\mathcal{O}(x_*,y_*;p_\perp)|y_\perp)
\end{aligned}
\tag{A.6}
$$

Note that the transverse arguments of all background fields in eq. (A.6) are effectively $y_\perp$.

### A.1.2 Scalar propagator for the complex conjugate amplitude

For calculations of the complex conjugate amplitude we need also the propagator

$$(x|\frac{1}{P^2-i\epsilon}|y) = \Big[i\theta(y_*-x_*)\int_0^\infty\frac{đ\alpha}{2\alpha} - i\theta(x_*-y_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]$$
$$\times e^{-i\alpha(x-y)_\bullet}(x_\perp|\mathrm{Pexp}\Big\{-i\int_{y_*}^{x_*}dz_*\Big[\frac{p_\perp^2}{\alpha s}-\frac{2g}{s}\tilde{A}_\bullet(z_*)\Big]\Big\}|y_\perp) \tag{A.7}$$

For the calculation of the square of Lipatov vertex we need to consider point $x$ inside the shock wave and point $y$ outside. In this case one should rewrite eq. (A.2) as follows

$$(x_\perp|\mathrm{Pexp}\Big\{-i\int_{y_*}^{x_*}dz_*\Big[\frac{p_\perp^2}{\alpha s}-\frac{2g}{s}\tilde{A}_\bullet(z_*)\Big]\Big\}|y_\perp)$$
$$= (x_\perp|\mathrm{Pexp}\Big\{\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ e^{-i\frac{p_\perp^2}{\alpha s}(x_*-z_*)}\tilde{A}_\bullet(z_*)e^{i\frac{p_\perp^2}{\alpha s}(x_*-z_*)}\Big\}e^{-i\frac{p_\perp^2}{\alpha s}(x_*-y_*)}|y_\perp) \tag{A.8}$$

The light-cone expansion around $x_\perp+\frac{2}{s}x_*p_1$ is given by eq. (A.3) with $y_*\to x_*$

$$e^{i\frac{p_\perp^2}{\alpha s}(z_*-x_*)}\tilde{A}_\bullet e^{-i\frac{p_\perp^2}{\alpha s}(z_*-x_*)}$$
$$= \tilde{A}_\bullet - \frac{z_*-x_*}{\alpha s}(2\tilde{F}_{\bullet j}p^j + i\tilde{D}^j\tilde{F}_{\bullet j}) - 2\frac{(z_*-x_*)^2}{\alpha^2s^2}(\tilde{D}_j\tilde{F}_{\bullet i}p^ip^j + i\tilde{D}^i\tilde{D}_j\tilde{F}_{\bullet i}p^j) + \ldots \tag{A.9}$$

(the only difference with the expansion (A.3) is that we should put the operators $p^j$ to the right) and therefore

$$\tilde{\mathcal{O}}(x_*,y_*;p_\perp) \equiv \mathrm{Pexp}\Big\{\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ e^{-i\frac{p_\perp^2}{\alpha s}(x_*-z_*)}\tilde{A}_\bullet(z_*)e^{i\frac{p_\perp^2}{\alpha s}(x_*-z_*)}\Big\} = 1$$
$$+\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ \Big[\tilde{A}_\bullet - \frac{z_*-x_*}{\alpha s}(2\tilde{F}_{\bullet j}p^j + i\tilde{D}^j\tilde{F}_{\bullet j}) - 2\frac{(z_*-x_*)^2}{\alpha^2s^2}(\tilde{D}_j\tilde{F}_{\bullet i}p^ip^j + i\tilde{D}^i\tilde{D}_j\tilde{F}_{\bullet i}p^j)\Big]$$
$$-\frac{4g^2}{s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\Big[\tilde{A}_\bullet - \frac{2(z-x)_*}{\alpha s}\tilde{F}_{\bullet i}p^i\Big]\Big[\tilde{A}_\bullet - \frac{2(z'-x)_*}{\alpha s}\tilde{F}_{\bullet j}p^j\Big] + \ldots \tag{A.10}$$

which turns to

$$\tilde{\mathcal{O}}(x_*,y_*;p_\perp) = [x_*,y_*] + \frac{2ig}{\alpha s^2}\int_{x_*}^{y_*}dz_*\ [x_*,z_*]\big\{2\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^j + i\tilde{D}^j\tilde{F}_{\bullet j}(z_*)[z_*,y_*]$$
$$+2\frac{(z-x)_*}{\alpha s}\big(\tilde{D}_k\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^jp^k + i\tilde{D}^j\tilde{D}_k\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^k\big)\big\}(z-x)_*$$
$$+\frac{8g^2}{\alpha s^3}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\ (z-x)_*[x_*,z_*]\Big(-i\tilde{F}_{\bullet j}(z_*)[z_*,z'_*]\tilde{F}_\bullet{}^j(z'_*)[z'_*,y_*]$$
$$-2\frac{(z'-x)_*}{\alpha s}\tilde{F}_{\bullet j}(z_*)[z_*,z'_*]\tilde{F}_{\bullet k}(z'_*)[z'_*,y_*]p^jp^k\Big) + \ldots \tag{A.11}$$

and we get

$$(x|\frac{1}{P^2-i\epsilon}|y) = \Big[i\theta(y_*-x_*)\int_0^\infty\frac{đ\alpha}{2\alpha} - i\theta(x_*-y_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]$$
$$\times\ e^{-i\alpha(x-y)_\bullet}(x_\perp|\tilde{\mathcal{O}}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|y_\perp) \tag{A.12}$$

Here (in eq. (A.11)) the transverse arguments of all background fields are effectively $x_\perp$.

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### A.1.3 The emission vertex

For the calculation of Lipatov vertex we need the propagator in mixed representation $(k|\frac{1}{P^2+i\epsilon}|z)$ in the limit $k^2\to 0$ where $k=\alpha p_1+\frac{k_\perp^2}{\alpha s}p_2+k_\perp$:

$$k^2(k|\frac{1}{P^2+i\epsilon}|y)=\frac{2}{s}\int dx_*dx_\bullet d^2x_\perp\ e^{ikx}\Big[-s\frac{\partial}{\partial x_\bullet}\frac{\partial}{\partial x_*}+\partial_\perp^2\Big](x|\frac{1}{P^2+i\epsilon}|y) \tag{A.13}$$

First, we perform the trivial integrations over $x_\bullet$ and $x_\perp$:

$$\begin{aligned}
&\lim_{k^2\to 0}k^2(k|\frac{1}{P^2+i\epsilon}|y) \qquad\qquad (A.14)\\
&=\!\int dx_*d^2x_\perp\ e^{i\frac{k_\perp^2}{\alpha s}x_*-i(k,x)_\perp}\Big[\frac{\partial}{\partial x_*}-\frac{i}{\alpha s}\partial_\perp^2\Big]\theta(x-y)_*(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\mathcal{O}(x_*,y_*;p_\perp)|y_\perp)e^{i\alpha y_\bullet}\\
&=\!\int dx_*d^2x_\perp\ e^{i\frac{k_\perp^2}{\alpha s}x_*-i(k,x)_\perp}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\frac{\partial}{\partial x_*}\theta(x-y)_*\mathcal{O}(x_*,y_*;p_\perp)|y_\perp)e^{i\alpha y_\bullet}\\
&=e^{i\alpha y_\bullet}e^{i\frac{k_\perp^2}{\alpha s}y_*}\!\int dx_*\frac{\partial}{\partial x_*}\theta(x-y)_*(k_\perp|\mathcal{O}(x_*,y_*;k)|y_\perp)\\
&=e^{i\alpha y_\bullet}e^{i\frac{k_\perp^2}{\alpha s}y_*-i(k,y)_\perp}\mathcal{O}(\infty,y_*,y_\perp;k)
\end{aligned}$$

where $\mathcal{O}(\infty,y_*,y_\perp;k)\equiv e^{i(k,y)_\perp}(k_\perp|\mathcal{O}(\infty,y_*;k)|y_\perp)$. In the explicit form

$$\begin{aligned}
&\mathcal{O}(\infty,y_*;y_\perp;k)\\
&=[\infty,y_*]_y-\frac{2ig}{\alpha s^2}\!\int_{y_*}^\infty\!dz_*\ (z-y)_*\Big(2k^j[\infty,z_*]_yF_{\bullet j}(z_*,y_\perp)-i[\infty,z_*]_yD^jF_{\bullet j}(z_*,y_\perp)\\
&\quad+2\frac{(z-y)_*}{\alpha s}(k^jk^l[\infty,z_*]_y-ik^l[\infty,z_*]_yD^j)D_lF_{\bullet j}(z_*,y_\perp)\Big)[z_*,y_*]_y\\
&\quad+\frac{8g^2}{\alpha s^3}\!\int_{y_*}^\infty\!dz_*\!\int_{y_*}^{z_*}\!dz'_*\ (z'-y)_*\Big(i[\infty,z_*]_yF_{\bullet j}(z_*,y_\perp)[z_*,z'_*]_yF_\bullet{}^j(z'_*,y_\perp)\\
&\quad-2k^jk^l\frac{(z-y)_*}{\alpha s}[\infty,z_*]_yF_{\bullet j}(z_*,y_\perp)[z_*,z'_*]_yF_{\bullet l}(z'_*,y_\perp)\Big)[z'_*,y_*]_y
\end{aligned} \tag{A.15}$$

where the transverse arguments of all fields are $y_\perp$ and $p^j$ is replaced by $k^j$.

Similarly, for the complex conjugate amplitude we get

$$\lim_{k^2\to 0}k^2(x|\frac{1}{P^2-i\epsilon}|k)=e^{-ikx}\tilde{\mathcal{O}}(x_*,\infty,x_\perp;k) \tag{A.16}$$

where $\tilde{\mathcal{O}}(x_*,\infty,x_\perp;k)\equiv(x_\perp|\tilde{\mathcal{O}}(x_*,\infty,p_\perp)|k_\perp)e^{-i(k,x)_\perp}$, or, in the explicit form

$$\begin{aligned}
&\tilde{\mathcal{O}}(x_*,\infty,x_\perp;k)\\
&=[x_*,\infty]_x+\frac{2ig}{\alpha s^2}\!\int_{x_*}^\infty\!dz_*\ [x_*,z_*]_x\big\{2\tilde F_{\bullet j}(z_*,x_\perp)[z_*,\infty]_xk^j+i\tilde D^j\tilde F_{\bullet j}(z_*,x_\perp)[z_*,\infty]_x\\
&\quad+2\frac{(z-x)_*}{\alpha s}\big(\tilde D_l\tilde F_{\bullet j}(z_*,x_\perp)[z_*,\infty]_xk^jk^l+i\tilde D^j\tilde D_l\tilde F_{\bullet j}(z_*,x_\perp)[z_*,\infty]_xk^l\big)\big\}(z-x)_*\\
&\quad+\frac{8g^2}{\alpha s^3}\!\int_{x_*}^\infty\!dz_*\!\int_{z_*}^\infty\!dz'_*\ (z-x)_*[x_*,z_*]_x\Big(-i\tilde F_{\bullet j}(z_*,x_\perp)[z_*,z'_*]_x\tilde F_\bullet{}^j(z'_*,x_\perp)[z'_*,\infty]_x\\
&\quad-2\frac{(z'-x)_*}{\alpha s}\tilde F_{\bullet j}(z_*,x_\perp)[z_*,z'_*]_x\tilde F_{\bullet l}(z'_*,x_\perp)[z'_*,\infty]_xk^jk^l\Big)+\dots
\end{aligned} \tag{A.17}$$

In the complex conjugate amplitude we expand around the light cone $x_\perp + \frac{2}{s}x_* p_1$ so the transverse arguments of all fields in eq. (A.17) are $x_\perp$. Note that the second terms in the r.h.s. of eqs. (A.6) and (A.7) (proportional to $\int_{-\infty}^0 {\textit{đ}}\alpha$) do not contribute since $\alpha > 0$ for the emitted particle.

## A.2 Gluon propagator

### A.2.1 Gluon propagator in the background gluon field

As we saw in the previous section, to get the emission vertex (A.13) it is sufficient to write down the propagator at $x_* > y_*$. The gluon propagator in the bF gauge has the form

$$
\begin{aligned}
i\langle A^a_\mu(x)A^b_\nu(y)\rangle &= (x|\frac{1}{P^2+2igF+i\epsilon}|y)^{ab}_{\mu\nu} \qquad\qquad \text{(A.18)}\\
&\overset{x_*>y_*}{=} -i\int_0^\infty \frac{{\textit{đ}}\alpha}{2\alpha}\; e^{-i\alpha(x-y)_\bullet}\;(x_\perp|\mathrm{Pexp}\Big\{-i\int_{y_*}^{x_*} dz_*\Big[\frac{p_\perp^2}{\alpha s} - \frac{2g}{s}A_\bullet(z_*) - \frac{2ig}{\alpha s}F(z_*)\Big]\Big\}|y_\perp)^{ab}_{\mu\nu}\\
&= -i\int_0^\infty \frac{{\textit{đ}}\alpha}{2\alpha}\; e^{-i\alpha(x-y)_\bullet}\\
&\quad\times(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x_*-y_*)}\mathrm{Pexp}\Big\{ig\int_{y_*}^{x_*} d\frac{2}{s}z_*\; e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big(A_\bullet + \frac{i}{\alpha}F\Big)(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big\}^{ab}_{\mu\nu}|y_\perp)
\end{aligned}
$$

where powers of $F$ are treated as usual, for example $(FA_\bullet FF)_{\mu\nu} \equiv F_\mu{}^\xi A_\bullet g_{\xi\eta}F^{\eta\lambda}F_{\lambda\nu}$. The expansion (A.3) now looks like

$$
\begin{aligned}
&e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big(A_\bullet g_{\mu\nu} + \frac{i}{\alpha}F_{\mu\nu}\Big)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)} = A_\bullet g_{\mu\nu} + \frac{i}{\alpha}F_{\mu\nu}\\
&+i\frac{z_*-y_*}{\alpha s}\Big[p_\perp^2, A_\bullet g_{\mu\nu} + \frac{i}{\alpha}F_{\mu\nu}\Big] - \frac{(z-y)_*^2}{2\alpha^2 s^2}\Big[p_\perp^2,\Big[p_\perp^2, A_\bullet g_{\mu\nu} + \frac{i}{\alpha}F_{\mu\nu}\Big]\Big] + \dots\\
&\quad= g_{\mu\nu}\Big[A_\bullet - \frac{z_*-y_*}{\alpha s}(2p^jF_{\bullet j} - iD^jF_{\bullet j}) - 2\frac{(z-y)_*^2}{\alpha^2 s^2}(p^jp^kD_jF_{\bullet k} - ip^kD_kD^jF_{\bullet j})\Big]\\
&\qquad+\frac{i}{\alpha}F_{\mu\nu} + 2i\frac{z_*-y_*}{\alpha^2 s}p^jD_jF_{\mu\nu} + 2i\frac{(z-y)_*^2}{\alpha^3 s^2}p^jp^kD_jD_kF_{\mu\nu} + \dots \qquad\qquad \text{(A.19)}
\end{aligned}
$$

so we get

$$
\begin{aligned}
&\mathcal{G}_{\mu\nu}(x_*,y_*;p_\perp) \equiv \mathrm{Pexp}\Big\{ig\int_{y_*}^{x_*} d\frac{2}{s}z_*\; e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big(A_\bullet + \frac{i}{\alpha}F\Big)(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big\}_{\mu\nu} \qquad\qquad \text{(A.20)}\\
&= g_{\mu\nu} + ig\int_{y_*}^{x_*} d\frac{2}{s}z_*\; e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big(A_\bullet + \frac{i}{\alpha}F\Big)_{\mu\nu}(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)} - g^2\int_{y_*}^{x_*} d\frac{2}{s}z_*\; e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\\
&\times\Big(A_\bullet + \frac{i}{\alpha}F\Big)_{\mu\xi}(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\int_{y_*}^{z_*} d\frac{2}{s}z'_*\; e^{i\frac{p_\perp^2}{\alpha s}(z'_*-y_*)}\Big(A_\bullet + \frac{i}{\alpha}F\Big)^\xi{}_\nu(z'_*)e^{-i\frac{p_\perp^2}{\alpha s}(z'_*-y_*)}\\
&= g_{\mu\nu} + \frac{2ig}{s}\int_{y_*}^{x_*} dz_*\Big\{g_{\mu\nu}\Big[A_\bullet(z_*) - \frac{(z-y)_*}{\alpha s}(2p^iF_{\bullet i} - iD^iF_{\bullet i}) - 2\frac{(z-y)_*^2}{\alpha^2 s^2}(p^jp^kD_jF_{\bullet k}\\
&\quad - ip^kD_kD^jF_{\bullet j})\Big] + \frac{i}{\alpha}F_{\mu\nu} + 2i\frac{z_*-y_*}{\alpha^2 s}p^jD_jF_{\mu\nu} + 2i\frac{(z-y)_*^2}{\alpha^3 s^2}p^jp^kD_jD_kF_{\mu\nu}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{4g^2}{s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\Big[g_{\mu\xi}A_\bullet+\frac{i}{\alpha}F_{\mu\xi}-2g_{\mu\xi}p^j\frac{(z-y)_*}{\alpha s}F_{\bullet j}\Big]\\
&\times\Big[\delta^\xi_\nu A_\bullet+\frac{i}{\alpha}F^\xi{}_\nu-2p^k\frac{(z'-y)_*}{\alpha s}F_{\bullet k}\delta^\xi_\nu\Big]+\dots\\
=\ &g_{\mu\nu}[x_*,y_*]+g\int_{y_*}^{x_*}dz_*\ \Big(-\frac{2i}{\alpha s^2}(z-y)_*g_{\mu\nu}\Big\{2p^j[x_*,z_*]F_{\bullet j}(z_*)-i[x_*,z_*]D^jF_{\bullet j}(z_*)\\
&+2\frac{(z-y)_*}{\alpha s}(p^jp^k[x_*,z_*]D_jF_{\bullet k}-ip^k[x_*,z_*]D_kD^jF_{\bullet j})\Big\}\\
&+\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\Big\{[x_*,z_*]F_{\bullet j}(z_*)+\frac{2(z-y)_*}{\alpha s}p^k[x_*,z_*]D_kF_{\bullet j}(z_*)\\
&+2\frac{(z-y)^2_*}{\alpha^2s^2}p^kp^l[x_*,z_*]D_kD_lF_{\bullet j}\Big\}\Big)[z_*,y_*]\\
&+\frac{8g^2}{\alpha s^3}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\Big(\Big[ig_{\mu\nu}(z'-y)_*-\frac{2}{\alpha s}p_{2\mu}p_{2\nu}\Big][x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_\bullet{}^j(z'_*)\\
&-2\frac{g_{\mu\nu}}{\alpha s}p^jp^k(z-y)_*(z'-y)_*[x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_{\bullet k}(z'_*)\Big)[z'_*,y_*]
\end{aligned}
$$

Note that $F_{\mu\xi}F^{\xi\eta}F_{\eta\nu}$ and higher terms of the expansion in powers of $F_{\mu\nu}$ vanish since the only non-vanishing field strength is $F_{\bullet i}$.

Finally,

$$
\begin{aligned}
(x|\frac{1}{P^2+2igF+i\epsilon}|y)^{ab}_{\mu\nu}\ =\ &\Big[-i\theta(x_*-y_*)\int_0^\infty\frac{đ\alpha}{2\alpha}+i\theta(y_*-x_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]\\
&\times\ e^{-i\alpha(x-y)_\bullet}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\mathcal{G}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)|y_\perp)
\end{aligned}
\tag{A.21}
$$

For the complex conjugate amplitude we obtain in a similar way

$$
\begin{aligned}
(x|\frac{1}{P^2+2igF-i\epsilon}|y)^{ab}_{\mu\nu}\ =\ &\Big[i\theta(y_*-x_*)\int_0^\infty\frac{đ\alpha}{2\alpha}-i\theta(x_*-y_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]\\
&\times\ e^{-i\alpha(x-y)_\bullet}(x_\perp|\tilde{\mathcal{G}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|y_\perp)
\end{aligned}
\tag{A.22}
$$

where

$$
\tilde{\mathcal{G}}_{\mu\nu}(x_*,y_*;p_\perp)\equiv\mathrm{Pexp}\Big\{ig\int_{y_*}^{x_*}d\frac{2}{s}z_*\ e^{i\frac{p_\perp^2}{\alpha s}(z_*-x_*)}\Big(\tilde{A}_\bullet+\frac{i}{\alpha}\tilde{F}\Big)(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-x_*)}\Big\}_{\mu\nu}
\tag{A.23}
$$

$$
\begin{aligned}
=\ &g_{\mu\nu}[x_*,y_*]+g\int_{x_*}^{y_*}dz_*\ [x_*,z_*]\Big(\frac{2i}{\alpha s^2}(z-x)_*g_{\mu\nu}\Big\{2\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^j+i\tilde{D}^j\tilde{F}_{\bullet j}(z_*)[z_*,y_*]\\
&+2\frac{(z-x)_*}{\alpha s}(\tilde{D}_j\tilde{F}_{\bullet k}(z_*)[z_*,y_*]p^jp^k+i\tilde{D}_k\tilde{D}^j\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^k)\Big\}\\
&-\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\Big\{\tilde{F}_{\bullet j}(z_*)[z_*,y_*]+\frac{2(z-x)_*}{\alpha s}\tilde{D}_k\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^k\\
&+2\frac{(z-x)^2_*}{\alpha^2s^2}\tilde{D}_k\tilde{D}_l\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^kp^l\Big\}\Big)
\end{aligned}
$$

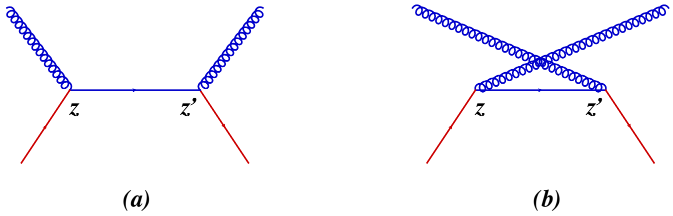


**Figure 7**. Gluon propagator in an external quark field.

$$
\begin{aligned}
&+\frac{8g^2}{\alpha s^3}\int_{x_*}^{y_*}dz_*\int_{z_*}^{y_*}dz'_*[x_*,z_*]\bigg(\bigg[-ig_{\mu\nu}(z-x)_*-\frac{2}{\alpha s}p_{2\mu}p_{2\nu}\bigg]\tilde{F}_{\bullet j}(z_*)[z_*,z'_*]\tilde{F}_{\bullet}{}^{j}(z'_*)[z'_*,y_*]\\
&-\frac{2g_{\mu\nu}}{\alpha s}(z-x)_*(z'-x)_*\tilde{F}_{\bullet j}(z_*)[z_*,z'_*]\tilde{F}_{\bullet k}(z'_*)[z'_*,y_*]p^jp^k\bigg)
\end{aligned}
$$

### A.2.2 Gluon propagator in the background quark field

We do not impose the condition $D^iF_{\bullet i}=0$ so our external field has quark sources $D^iF^a_{\bullet i}=g\bar\psi t^a\not{p}_1\psi$ which we need to take into consideration. The corresponding contribution to gluon propagator comes from diagrams in figure 7

$$
\begin{aligned}
\langle A^a_\mu(x)A^b_\nu(y)\rangle_{\text{figure 7}} = g^2\int d^4zd^4z'(x|\frac{1}{P^2+i\epsilon}|z)^{ac}(z'|\frac{1}{P^2+i\epsilon}|y)^{db}\\
\times\bigg[(z|\bar\psi t^c\gamma_\mu\not{P}\frac{i}{\not{P}^2+i\epsilon}\gamma_\nu t^d\psi|z')+(z'|\bar\psi t^d\gamma_\nu\frac{i}{\not{P}^2+i\epsilon}\not{P}\gamma_\mu t^c\psi|z)\bigg]
\end{aligned}
\tag{A.24}
$$

As we mentioned above, we can consider quark fields with $+\frac{1}{2}$ spin projection onto $p_1$ direction which corresponds to $\bar\psi(\ldots)\psi$ operators of leading collinear twist. In this approximation $\not{p}_2\psi=0$ so the only non-zero propagators are $\langle A_\bullet(x)A_\bullet(y)\rangle$, $\langle A_\bullet(x)A_i(y)\rangle$ and $\langle A_i(x)A_j(y)\rangle$. In addition, we assume that the quark fields $\psi(z)$ depend only on $z_\perp$ and $z_*$ (same as gluon fields) so the operator $\hat\alpha=\frac{2}{s}\hat{p}_*$ commutes with all background-field operators. We get

$$
\begin{aligned}
\langle A^a_\bullet(x)A^b_\bullet(y)\rangle_{\text{figure 7}} = 2ig^2\int d^4zd^4z'\ (x|\frac{1}{P^2+i\epsilon}|z)^{ac}(z'|\frac{1}{P^2+i\epsilon}|y)^{db}\\
\times\bigg[(z|\bar\psi t^c\not{p}_1P_\bullet\frac{1}{\not{P}^2+i\epsilon}\ t^d\psi|z')+(z'|\bar\psi t^d\frac{1}{\not{P}^2+i\epsilon}P_\bullet\not{p}_1t^c\psi|z)\bigg]
\end{aligned}
\tag{A.25}
$$

In our gluon field $\not{P}=\alpha\not{p}_1+2\frac{\not{p}_2}{s}P_\bullet+\not{p}_\perp$ so $\not{P}^2=2\alpha P_\bullet-p_\perp^2+\frac{2i}{s}\not{p}_2\gamma^jF_{\bullet j}$ and one can rewrite $P_\bullet\frac{i}{\not{P}^2+i\epsilon}$ as

$$
\begin{aligned}
P_\bullet\frac{1}{2\alpha P_\bullet-p_\perp^2+i\epsilon} &= \bigg[\frac{p_\perp^2}{2\alpha}+\frac{1}{2\alpha}(2\alpha P_\bullet-p_\perp^2)\bigg]\frac{1}{2\alpha P_\bullet-p_\perp^2+i\epsilon}\\
&= \frac{1}{2\alpha}+\frac{p_\perp^2}{2\alpha}\frac{1}{2\alpha P_\bullet-p_\perp^2+i\epsilon}
\end{aligned}
\tag{A.26}
$$

(the term $\frac{2}{s}\not{p}_2\gamma^jF_{\bullet j}$ does not contribute due to $\not{p}_2\psi=0$). Similarly,

$$
\frac{1}{2\alpha P_\bullet-p_\perp^2+i\epsilon}P_\bullet=\frac{1}{2\alpha}+\frac{1}{2\alpha P_\bullet-p_\perp^2+i\epsilon}\frac{p_\perp^2}{2\alpha}
\tag{A.27}
$$

so one can rewrite the propagator (A.24) as

$$\begin{aligned}\langle A^a_\bullet(x)A^b_\bullet(y)\rangle_{\text{figure 7}} &= -ig(x|\frac{1}{\alpha(P^2+i\epsilon)}D^jF_{\bullet j}\frac{1}{P^2+i\epsilon}|y)^{ab} \\ &+ig^2(x|\left(\frac{1}{\alpha(P^2+i\epsilon)}\right)^{ac}(p_\perp^2\bar\psi+2ip^j\partial_j\bar\psi-\partial_\perp^2\bar\psi)t^c\frac{1}{P^2+i\epsilon}\not{p}_1 t^d\psi\left(\frac{1}{P^2+i\epsilon}\right)^{db}|y) \\ &+ig^2(y|\left(\frac{1}{\alpha(P^2+i\epsilon)}\right)^{bd}\bar\psi t^d\frac{1}{P^2+i\epsilon}t^c\not{p}_1(\psi p_\perp^2-2i\partial_j\psi p^j-\partial_\perp^2\psi)\left(\frac{1}{P^2+i\epsilon}\right)^{ca}|x)\end{aligned} \tag{A.28}$$

where in the first line we have rewritten $g^2\bar\psi[t^c,t^d]\not{p}_1\psi$ as $-g(D^jF_{\bullet j})^{cd}$.

Similarly, we get

$$\begin{aligned}&\langle A^a_\bullet(x)A^b_i(y)\rangle_{\text{figure 7}} \\ &\quad= -ig^2(x|\left(\frac{1}{P^2+i\epsilon}\right)^{ac}(p^j\bar\psi-i\partial^j\bar\psi)\gamma_j\not{p}_1\gamma_i t^c\frac{1}{P^2+i\epsilon}t^d\psi\left(\frac{1}{P^2+i\epsilon}\right)^{db}|y) \\ &\qquad-ig^2(y|\left(\frac{1}{P^2+i\epsilon}\right)^{bd}\bar\psi t^d\frac{1}{P^2+i\epsilon}t^c\gamma_i\not{p}_1\gamma_j(\psi p^j+i\partial^j\psi)\left(\frac{1}{P^2+i\epsilon}\right)^{ca}|x), \\ &\langle A^a_i(x)A^b_j(y)\rangle_{\text{figure 7}} = ig^2(x|\left(\frac{\alpha}{P^2+i\epsilon}\right)^{ac}\bar\psi\gamma_i\not{p}_1\gamma_j t^c\frac{1}{P^2+i\epsilon}t^d\psi\left(\frac{1}{P^2+i\epsilon}\right)^{db}|y) \\ &\qquad\qquad+ig^2(y|\left(\frac{\alpha}{P^2+i\epsilon}\right)^{bd}\bar\psi t^d\frac{1}{P^2+i\epsilon}t^c\gamma_j\not{p}_1\gamma_i\psi\left(\frac{1}{P^2+i\epsilon}\right)^{ca}|x)\end{aligned} \tag{A.29}$$

for the remaining propagators.

If now the point $y$ lies inside the shock wave we can expand the gluon and quark propagators around the light ray $y_\perp+\frac{2}{s}y_*p_1$. It is easy to see that the expansion of the gluon fields $A_\bullet$ given by eq. (A.3) exceeds our twist-two accuracy so we need only expansion of quark fields which is

$$e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\psi e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)} = \psi+2\frac{z_*-y_*}{\alpha s}p^jD_j\psi+\dots \tag{A.30}$$

(and similarly for $\bar\psi$ and $D^iF_{\bullet i}$).

It is convenient to parametrize quark contribution in the same way as the gluon one (A.21)

$$\begin{aligned}\langle A^a_\mu(x)A^b_\nu(y)\rangle_{\text{figure 7}} &= \left[-\theta(x_*-y_*)\int_0^\infty\frac{d\!\!{}^-\alpha}{2\alpha}+\theta(y_*-x_*)\int_{-\infty}^0\frac{d\!\!{}^-\alpha}{2\alpha}\right]e^{-i\alpha(x-y)_\bullet} \\ &\times\left[(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\mathcal{Q}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)|y_\perp)+(y_\perp|\bar{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|x_\perp)\right]\end{aligned} \tag{A.31}$$

In the leading order we need only the first two terms of the expansion (A.30) which gives

$$
\begin{aligned}
&\mathcal{Q}^{ab}_{\bullet\bullet}(x_*,y_*;p_\perp)\\
&=-\frac{ig}{\alpha^2 s}\int_{y_*}^{x_*}dz_*\Big([x_*,z_*]D^jF_{\bullet j}(z_*)[z_*,y_*]+\frac{2(z-y)_*}{\alpha s}p^k[x_*,z_*]D_kD^jF_{\bullet j}[z_*,y_*]\Big)^{ab}\\
&\quad+\frac{g^2}{\alpha^3 s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\ \Big[\big(p_\perp^2\bar\psi(z_*)+2ip^j\bar\psi\overleftarrow{D}_j(z_*)\big)\not{p}_1[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\psi(z'_*)\\
&\quad+2\frac{z_*-y_*}{\alpha s}p_\perp^2p^j\bar\psi\overleftarrow{D}_j(z_*)\not{p}_1[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\psi(z'_*)\\
&\quad+2\frac{z'_*-y_*}{\alpha s}p_\perp^2p^j\bar\psi(z_*)\not{p}_1[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]D_j\psi(z'_*)\Big]
\end{aligned}
\tag{A.32}
$$

and

$$
\begin{aligned}
&\bar{\mathcal{Q}}^{ab}_{\bullet\bullet}(x_*,y_*;p_\perp)\\
&=-\frac{g^2}{\alpha^3 s^2}\int_{y_*}^{x_*}dz_*\int_{z_*}^{x_*}dz'_*\Big[\bar\psi(z_*)\not{p}_1[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\big(\psi(z'_*)p_\perp^2-2iD_j\psi(z'_*)p^j\big)\\
&\quad-2\frac{z_*-y_*}{\alpha s}\bar\psi\overleftarrow{D}_j(z_*)\not{p}_1[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\psi(z'_*)p_\perp^2p^j\\
&\quad-2\frac{z'_*-y_*}{\alpha s}\bar\psi(z_*)\not{p}_1[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]D_j\psi(z'_*)p_\perp^2p^j\Big]
\end{aligned}
\tag{A.33}
$$

Similarly, for propagator $\langle A_\bullet(x)A_i(y)\rangle$ one gets

$$
\begin{aligned}
&\mathcal{Q}^{ab}_{\bullet i}(x_*,y_*;p_\perp)\\
&=-\frac{g^2}{\alpha^2 s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\Big[[p^j\bar\psi(z_*)-i\bar\psi\overleftarrow{D}^j(z_*)]\gamma_j\not{p}_1\gamma_i[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\psi(z'_*)\\
&\quad+\frac{2(z-y)_*}{\alpha s}p^jp^k\bar\psi\overleftarrow{D}_k(z_*)\gamma_j\not{p}_1\gamma_i[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\psi(z'_*)\\
&\quad+\frac{2(z'-y)_*}{\alpha s}p^jp^k\bar\psi\gamma_j\not{p}_1\gamma_i[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]D_k\psi(z'_*)\Big]
\end{aligned}
\tag{A.34}
$$

$$
\begin{aligned}
&\bar{\mathcal{Q}}^{ab}_{\bullet i}(x_*,y_*;p_\perp)\\
&=-\frac{g^2}{\alpha^2 s^2}\int_{y_*}^{x_*}dz_*\int_{z_*}^{x_*}dz'_*\ \Big[\bar\psi(z_*)\gamma_i\not{p}_1\gamma_j[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*](\psi p^j+iD^j\psi)(z'_*)\\
&\quad-2\frac{z_*-y_*}{\alpha s}\bar\psi\overleftarrow{D}_k(z_*)\gamma_i\not{p}_1\gamma_j[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\psi(z'_*)p^jp^k\\
&\quad-2\frac{z'_*-y_*}{\alpha s}\bar\psi(z_*)\gamma_i\not{p}_1\gamma_j[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]D_k\psi(z'_*)p^jp^k\Big]
\end{aligned}
\tag{A.35}
$$

For the propagator $\langle A_i(x)A_\bullet(y)\rangle$ the corresponding expressions $\mathcal{Q}^{ab}_{i\bullet}(x_*,y_*;p_\perp)$ and $\bar{\mathcal{Q}}^{ab}_{i\bullet}(x_*,y_*;p_\perp)$ are obtained from eqs. (A.34) and (A.35) by replacements in the r.h.s.'s $\gamma_j\not{p}_1\gamma_i\to\gamma_i\not{p}_1\gamma_j$ and $\gamma_i\not{p}_1\gamma_j\to\gamma_j\not{p}_1\gamma_i$, respectively.



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Finally, for the propagator $\langle A_i(x)A_j(y)\rangle$ we obtain

$$\begin{aligned}
\mathcal{Q}^{ab}_{ij}(x_*,y_*;p_\perp) \;=\; & \frac{g^2}{\alpha s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\Big[\bar\psi(z_*)\gamma_i\,\not{p}_1\gamma_j[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\psi(z'_*)\\
&+\frac{2(z-y)_*}{\alpha s}p^k\bar\psi\overleftarrow{D}_k\,(z_*)\gamma_i\,\not{p}_1\gamma_j[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\psi(z'_*)\\
&+\frac{2(z'-y)_*}{\alpha s}p^k\bar\psi\gamma_i\,\not{p}_1\gamma_j[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]D_k\psi(z'_*)\Big] \qquad \text{(A.36)}
\end{aligned}$$

$$\begin{aligned}
\bar{\mathcal{Q}}^{ab}_{ij}(x_*,y_*;p_\perp) \;=\; & -\frac{g^2}{\alpha s^2}\int_{y_*}^{x_*}dz_*\int_{z_*}^{x_*}dz'_*\;\Big[\bar\psi(z_*)\gamma_j\,\not{p}_1\gamma_i[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\psi(z'_*)\\
&-2\frac{z_*-y_*}{\alpha s}\bar\psi(z_*)\overleftarrow{D}_k\;\gamma_j\,\not{p}_1\gamma_i[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\psi(z'_*)p^k\\
&-2\frac{z'_*-y_*}{\alpha s}\bar\psi(z_*)\gamma_j\,\not{p}_1\gamma_i[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]D_k\psi(z'_*)p^k\Big] \qquad \text{(A.37)}
\end{aligned}$$

For the complex conjugate amplitude we get in a similar way

$$\begin{aligned}
\langle\tilde A^a_\mu(x)\tilde A^b_\nu(y)\rangle_{\text{figure 7}} = & \Big[-\theta(y_*-x_*)\int_0^\infty\frac{đ\alpha}{2\alpha}+\theta(x_*-y_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]\;e^{-i\alpha(x-y)_\bullet}\\
\times\Big[&(x_\perp|\tilde{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|y_\perp)+(y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\bar{\tilde{\mathcal{Q}}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)|x_\perp)\Big] \qquad \text{(A.38)}
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{\mathcal{Q}}^{ab}_{\bullet\bullet}(x_*,y_*;p_\perp)\\
&=\frac{-ig}{\alpha^2 s}\int_{y_*}^{x_*}dz_*\Big([x_*,z_*]\tilde D^j\tilde F_{\bullet j}(z_*)[z_*,y_*]-\frac{2(x-z)_*}{\alpha s}[x_*,z_*]\tilde D_k\tilde D^j\tilde F_{\bullet j}[z_*,y_*]p^k\Big)^{ab}\\
&\quad+\frac{g^2}{\alpha^3 s^2}\int_{x_*}^{y_*}dz_*\int_{z_*}^{y_*}dz'_*\Big[\bar{\tilde\psi}(z_*)\,\not{p}_1[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\big(\tilde\psi(z'_*)p_\perp^2-2i\tilde D_j\tilde\psi(z'_*)p^j\big)\\
&\quad+2\frac{z_*-x_*}{\alpha s}\bar{\tilde\psi}\overleftarrow{\tilde D}_j\;(z_*)\,\not{p}_1[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde\psi(z'_*)p_\perp^2p^j\\
&\quad+2\frac{z'_*-x_*}{\alpha s}\bar{\tilde\psi}(z_*)\,\not{p}_1[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde D_j\tilde\psi(z'_*)p_\perp^2p^j \qquad \text{(A.39)}
\end{aligned}$$

$$\begin{aligned}
&\bar{\tilde{\mathcal{Q}}}^{ab}_{\bullet\bullet}(x_*,y_*;p_\perp)\\
&=-\frac{g^2}{\alpha^3 s^2}\int_{x_*}^{y_*}dz_*\int_{x_*}^{z_*}dz'_*\;\Big[\big(p_\perp^2\bar{\tilde\psi}(z_*)+2ip^j\bar{\tilde\psi}\overleftarrow{\tilde D}_j\,(z_*)\big)\,\not{p}_1[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde\psi(z'_*)\\
&\quad-2\frac{z_*-x_*}{\alpha s}p_\perp^2p^j\bar{\tilde\psi}\overleftarrow{\tilde D}_j\;(z_*)\,\not{p}_1[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde\psi(z'_*)\\
&\quad-2\frac{z'_*-x_*}{\alpha s}p_\perp^2p^j\bar{\tilde\psi}(z_*)\,\not{p}_1[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde D_j\tilde\psi(z'_*)\Big] \qquad \text{(A.40)}
\end{aligned}$$

and

$$\tilde{\mathcal{Q}}^{ab}_{\bullet i}(x_*,y_*;p_\perp)$$
$$= \frac{-g^2}{\alpha^2 s^2}\int_{x_*}^{y_*} dz_* \int_{z_*}^{y_*} dz'_* \Big[\bar{\tilde{\psi}}(z_*)\gamma_j \not{p}_1\gamma_i[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*](\tilde{\psi}p^j + i\tilde{D}^j\psi)$$
$$+2\frac{z_*-x_*}{\alpha s}\bar{\tilde{\psi}}\overleftarrow{\tilde{D}}_k(z_*)\gamma_j \not{p}_1\gamma_i[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde{\psi}(z'_*)p^j p^k$$
$$+2\frac{z'_*-x_*}{\alpha s}\bar{\tilde{\psi}}(z_*)\gamma_j \not{p}_1\gamma_i[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde{D}_k\tilde{\psi}(z'_*)p^j p^k\Big] \tag{A.41}$$

$$\bar{\tilde{\mathcal{Q}}}^{ab}_{\bullet i}(x_*,y_*;p_\perp)$$
$$= \frac{-g^2}{\alpha^2 s^2}\int_{x_*}^{y_*} dz_* \int_{x_*}^{z_*} dz'_* \Big[(p^j\bar{\tilde{\psi}} - i\tilde{D}^j\bar{\tilde{\psi}})\gamma_i \not{p}_1\gamma_j[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde{\psi}(z'_*)$$
$$-2\frac{z_*-x_*}{\alpha s}p^j p^k\bar{\tilde{\psi}}\overleftarrow{\tilde{D}}_k(z_*)\gamma_i \not{p}_1\gamma_j[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde{\psi}(z'_*)$$
$$-2\frac{z'_*-x_*}{\alpha s}p^j p^k\bar{\tilde{\psi}}(z_*)\gamma_i \not{p}_1\gamma_j[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde{D}_k\tilde{\psi}(z'_*)\Big] \tag{A.42}$$

To get $\tilde{\mathcal{Q}}^{ab}_{i\bullet}$ one should again make the replacement $\gamma_j \not{p}_1\gamma_i \to \gamma_i \not{p}_1\gamma_j$ in eq. (A.41) and to get $\bar{\tilde{\mathcal{Q}}}^{ab}_{i\bullet}$ the replacement $\gamma_i \not{p}_1\gamma_j \to \gamma_j \not{p}_1\gamma_i$ in eq. (A.42). Finally, similarly to eq. (A.36) one obtains

$$\tilde{\mathcal{Q}}^{ab}_{ij}(x_*,y_*;p_\perp) = \frac{g^2}{\alpha s^2}\int_{x_*}^{y_*} dz_* \int_{z_*}^{y_*} dz'_* \Big[\bar{\tilde{\psi}}(z_*)\gamma_i \not{p}_1\gamma_j[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde{\psi}$$
$$+2\frac{z_*-x_*}{\alpha s}\bar{\tilde{\psi}}\overleftarrow{\tilde{D}}_k(z_*)\gamma_i \not{p}_1\gamma_j[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde{\psi}(z'_*)p^k$$
$$+2\frac{z'_*-x_*}{\alpha s}\bar{\tilde{\psi}}(z_*)\gamma_i \not{p}_1\gamma_j[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde{D}_k\tilde{\psi}(z'_*)p^k\Big] \tag{A.43}$$

$$\bar{\tilde{\mathcal{Q}}}^{ab}_{ij}(x_*,y_*;p_\perp) = -\frac{g^2}{\alpha s^2}\int_{x_*}^{y_*} dz_* \int_{x_*}^{z_*} dz'_* \Big[\bar{\tilde{\psi}}\gamma_j \not{p}_1\gamma_i[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde{\psi}(z'_*)$$
$$-2\frac{z_*-x_*}{\alpha s}p^k\bar{\tilde{\psi}}\overleftarrow{\tilde{D}}_k(z_*)\gamma_j \not{p}_1\gamma_i[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde{\psi}(z'_*)$$
$$-2\frac{z'_*-x_*}{\alpha s}p^k\bar{\tilde{\psi}}(z_*)\gamma_j \not{p}_1\gamma_i[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde{D}_k\tilde{\psi}(z'_*)\Big] \tag{A.44}$$

#### A.2.3 Final form of the gluon propagator

Assembling terms from two previous sections we get the final result for background-Feynman gluon propagator in external field in the form

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle \tag{A.45}$$
$$= \Big[-\theta(x_*-y_*)\int_0^\infty \frac{d\!\!\bar{}\,\alpha}{2\alpha} + \theta(y_*-x_*)\int_{-\infty}^0 \frac{d\!\!\bar{}\,\alpha}{2\alpha}\Big]e^{-i\alpha(x-y)_\bullet}\big\{(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}$$
$$\times\big[\mathcal{G}^{ab}_{\mu\nu}(x_*,y_*;p_\perp) + \mathcal{Q}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)\big]|y_\perp) + (y_\perp|\bar{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|x_\perp)\big\}$$

for Feynman propagator and

$$\langle\tilde A^a_\mu(x)\tilde A^b_\nu(y)\rangle \tag{A.46}$$
$$=\Big[-\theta(y_*-x_*)\int_0^\infty\frac{đ\alpha}{2\alpha}+\theta(x_*-y_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]e^{-i\alpha(x-y)_\bullet}\big\{(x_\perp|\big[\tilde{\mathcal G}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)$$
$$+\tilde{\mathcal Q}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)\big]e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|y_\perp)+(y_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\bar{\tilde{\mathcal Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)|x_\perp)\big\}$$

for the anti-Feynman propagator in the complex conjugate amplitude.

### A.3 Vertex of gluon emission

Repeating the steps which lead us to eq. (A.14) we obtain

$$\lim_{k^2\to0}k^2\langle A^a_\mu(k)A^b_\nu(y)\rangle = -ie^{iky}\mathcal O^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k), \tag{A.47}$$
$$\mathcal O^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k) = \mathcal G^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k)+\mathcal Q^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k)+\bar{\mathcal Q}^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k)$$

where

$$\begin{aligned}\mathcal G^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k) &\equiv e^{i(k,y)_\perp}(k_\perp|\mathcal G^{ab}_{\mu\nu}(\infty,y_*;p_\perp)|y_\perp),\\ \mathcal Q^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k) &\equiv e^{i(k,y)_\perp}(k_\perp|\mathcal Q^{ab}_{\mu\nu}(\infty,y_*,p_\perp)|y_\perp),\\ \bar{\mathcal Q}^{ab}_{\mu\nu}(\infty,y_*,y_\perp;k) &\equiv e^{-i(k,y)_\perp}(y_\perp|\bar{\mathcal Q}^{ab}_{\mu\nu}(\infty,y_*;p_\perp)|k_\perp)\end{aligned} \tag{A.48}$$

The explicit expressions can be read from eqs. (A.20) and (A.32)–(A.37) by taking the transverse arguments of all fields to be $y_\perp$ and replacing the operators $p^j$ with $k^j$ similarly to eq. (A.15).

Similarly, for the complex conjugate amplitude the emission vertex takes the form

$$\lim_{k^2\to0}k^2\langle\tilde A^a_\mu(x)\tilde A^b_\nu(k)\rangle = ie^{-ikx}\tilde{\mathcal O}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k), \tag{A.49}$$
$$\tilde{\mathcal O}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k) = \tilde{\mathcal G}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k)+\tilde{\mathcal Q}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k)+\bar{\tilde{\mathcal Q}}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k)$$

where

$$\begin{aligned}\tilde{\mathcal G}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k) &\equiv e^{-i(k,x)_\perp}(x_\perp|\tilde{\mathcal G}^{ab}_{\mu\nu}(x_*,\infty;p_\perp)|k_\perp),\\ \tilde{\mathcal Q}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k) &\equiv e^{-i(k,x)_\perp}(x_\perp|\tilde{\mathcal Q}^{ab}_{\mu\nu}(x_*,\infty;p_\perp)|k_\perp),\\ \bar{\tilde{\mathcal Q}}^{ab}_{\mu\nu}(x_*,\infty,x_\perp;k) &\equiv e^{i(k,x)_\perp}(k_\perp|\bar{\tilde{\mathcal Q}}^{ab}_{\mu\nu}(x_*,\infty;p_\perp)|x_\perp)\end{aligned} \tag{A.50}$$

Again, the explicit expressions can be read from eqs. (A.23) and (A.39)–(A.44) by taking the transverse arguments of all fields to be $x_\perp$ (and replacing the operators $p^j$ with $k^j$) similarly to eq. (A.17).

## B Propagators in the shock-wave background

In this section we consider propagators of slow fields in the background of fast fields in the case when the characteristic transverse momenta of fast fields ($k_\perp$) and slow fields ($l_\perp$) are comparable. In this case the usual rescaling of ref. [8, 9] applies and we can again consider the external fields of the type $A_\bullet(x_*, x_\perp)$ with $A_i = A_* = 0$.

Actually, since the typical longitudinal size of fast fields is $\sigma_* \sim \frac{\sigma' s}{l_\perp^2}$ and the typical distances traveled by slow gluons are $\sim \frac{\sigma s}{k_\perp^2}$ our formulas will remain correct if $l_\perp^2 \gg k_\perp^2$ since the shock wave is even thinner in this case. As we discussed above, we assume that the support of the shock wave is thin but not infinitely thin. For our calculations we need gluon propagators with both points outside the shock wave and propagator with one point inside and one outside. It is convenient to start from the latter case since all the necessary formulas can be deduced from the light-cone expansion discussed in the previous section. To illustrate this, let us again for simplicity consider scalar propagator.

### B.1 Propagators with one point in the shock wave

#### B.1.1 Scalar propagator

For simplicity we will again perform at first the calculation for "scalar propagator" $(x|\frac{1}{P^2+i\epsilon}|y)$. As usual, we assume that the only nonzero component of the external field is $A_\bullet$ and it does not depend on $z_\bullet$ so the operator $\alpha = i\frac{\partial}{\partial z_\bullet}$ commutes with all background fields. The propagator in the external field $A_\bullet(z_*, z_\perp)$ is given by eq. (A.1) and (A.2) which can be rewritten as

$$(x|\frac{1}{P^2+i\epsilon}|y) = \Big[ -i\theta(x_* - y_*)\int_0^\infty \frac{đ\alpha}{2\alpha} + i\theta(y_* - x_*)\int_{-\infty}^0 \frac{đ\alpha}{2\alpha}\Big]\ e^{-i\alpha(x-y)_\bullet} \tag{B.1}$$
$$\times(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x_*-y_*)}\mathrm{Pexp}\Big\{\frac{2ig}{s}\int_{y_*}^{x_*}dz_*\ e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}A_\bullet(z_*)e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\Big\}|y_\perp)$$

Suppose the point $y$ lies inside the shock wave (the point $x$ may be inside or outside of the shock wave). Since the longitudinal distances $z_*$ inside the shock wave are small ($\sim \frac{\sigma' s}{l_\perp^2}$) we can use the expansion (A.3) but the parameter of the expansion is now $\frac{p_\perp^2}{\alpha s}\sigma_* \sim \frac{\sigma'}{\alpha} \ll 1$ rather than twist of the operator. Consequently, the last term in eq. (A.3) can be neglected since it has an extra factor $\frac{p_\perp^2}{\alpha s}\sigma_*$ in comparison to the second term:

$$\begin{aligned}
&e^{i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}A_\bullet e^{-i\frac{p_\perp^2}{\alpha s}(z_*-y_*)}\\
&\quad = A_\bullet - \frac{z_* - y_*}{\alpha s}(2p^iF_{\bullet i} - iD^iF_{\bullet i}) - 2\frac{(z_*-y_*)^2}{\alpha^2 s^2}(p^ip^j - ip^jD^i)D_jF_{\bullet i} + \dots\\
&\quad = A_\bullet - \frac{z_* - y_*}{\alpha s}(2p^iF_{\bullet i} - iD^iF_{\bullet i}) + \dots
\end{aligned} \tag{B.2}$$

This is again the expansion around the light ray $y_\perp + \frac{2}{s}y_*p_1$ but now with the parameter of the expansion $\sim \frac{p_\perp^2}{\alpha s}\sigma_* \ll 1$. However, we need to keep the second term of this expansion since the first term forms gauge links (for example, it is absent in the $A_\bullet = 0$ gauge).

Since there are no new terms in the expansion (B.2) in comparison to (A.3) we can look at the final result (A.5) for $\mathcal{O}(x_*, y_*; p_\perp)$ and drop the terms which are small with respect to our new power counting. This way the eq. (A.5) reduces to

$$\begin{aligned}
&\mathcal{O}(x_*,y_*;p_\perp)\\
&=[x_*,y_*]-\frac{2ig}{\alpha s^2}\int_{y_*}^{x_*}dz_*\ (z-y)_*\big\{2p^j[x_*,z_*]F_{\bullet j}(z_*)-i[x_*,z_*]D^jF_{\bullet j}(z_*)\big\}[z_*,y_*]\\
&\quad+\frac{8ig^2}{\alpha s^3}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\ (z'-y)_*[x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_\bullet{}^j(z'_*)[z'_*,y_*]+\dots
\end{aligned}\tag{B.3}$$

and the propagator has the form (A.6)

$$\begin{aligned}
(x|\frac{1}{P^2+i\epsilon}|y)=\Big[-i\theta(x_*-y_*)\int_0^\infty\frac{đ\alpha}{2\alpha}+i\theta(y_*-x_*)\int_{-\infty}^0\frac{đ\alpha}{2\alpha}\Big]\\
\times e^{-i\alpha(x-y)_\bullet}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\mathcal{O}(x_*,y_*;p_\perp)|y_\perp)
\end{aligned}\tag{B.4}$$

As we mentioned, this formula is correct for the point $y$ inside the shock wave and the point $x$ inside or outside.

Similarly, for the complex conjugate amplitude we obtain the propagator in the form (A.12) with

$$\begin{aligned}
&\tilde{\mathcal{O}}(x_*,y_*;p_\perp)\\
&=[x_*,y_*]+\frac{2ig}{\alpha s^2}\int_{x_*}^{y_*}dz_*\ (z-x)_*\ [x_*,z_*]\big\{2\tilde F_{\bullet j}(z_*)[z_*,y_*]p^j+i\tilde D^j\tilde F_{\bullet j}(z_*)[z_*,y_*]\big\}\\
&\quad-\frac{8ig^2}{\alpha s^3}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\ (z-x)_*[x_*,z_*]\tilde F_{\bullet j}(z_*)[z_*,z'_*]\tilde F_\bullet{}^j(z'_*)[z'_*,y_*]+\dots
\end{aligned}\tag{B.5}$$

which is the expansion (A.11) but with fewer number of terms. Again, the formula (A.12) with $\tilde{\mathcal{O}}(x_*,y_*;p_\perp)$ given by the above expression is correct for the point $x$ inside the shock wave and the point $y$ inside or outside.

The expressions for particle production are the same as (A.14) and (A.16) with $\mathcal{O}(\infty,y_*,y_\perp;k)$ and $\tilde{\mathcal{O}}(x_*,\infty,x_\perp;k)$ changed to eqs. (B.3) and (B.5), respectively.

### B.1.2 Gluon propagator and vertex of gluon emission

As we saw in previous section, the gluon propagator with one point in the shock wave can be obtained in the same way as the propagator near the light cone, only the parameter of the expansion is different: $\frac{p_\perp^2}{\alpha s}\sigma_*$ rather than the twist of the operator. Careful inspection of the expansions (A.19) and (A.30) reveals that there is no leading or next-to-leading terms with twist larger than four so we can recycle the final formulas (A.45) and (A.46) for gluon propagators. At $\frac{p_\perp^2}{\alpha s}\sigma_*\ll 1$ the expression (A.20) for $\mathcal{G}_{\mu\nu}(x_*,y_*;p_\perp)$ turns to

$$\begin{aligned}
&\mathcal{G}_{\mu\nu}(x_*,y_*;p_\perp)\\
&=g_{\mu\nu}[x_*,y_*]+g\int_{y_*}^{x_*}dz_*\ \Big(-\frac{2i}{\alpha s^2}(z-y)_*g_{\mu\nu}\big\{2p^j[x_*,z_*]F_{\bullet j}(z_*)-i[x_*,z_*]D^jF_{\bullet j}(z_*)\big\}\\
&\quad+\frac{4}{\alpha s^2}(\delta_\mu^jp_{2\nu}-\delta_\nu^jp_{2\mu})[x_*,z_*]F_{\bullet j}(z_*)\Big)[z_*,y_*]\\
&\quad+\frac{8g^2}{\alpha s^3}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\ \Big[ig_{\mu\nu}(z'-y)_*-\frac{2}{\alpha s}p_{2\mu}p_{2\nu}\Big][x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_\bullet{}^j(z'_*)[z'_*,y_*]
\end{aligned}\tag{B.6}$$

Looking at quark formulas (A.32)–(A.37) we see that at $\frac{p_\perp^2}{\alpha s}\sigma_* \ll 1$ the only surviving terms are the first terms in the r.h.s's of these equations. Let us compare now the size of these terms to the gluon contribution (B.6). The "power counting" for external quark fields in comparison to gluon ones is $\frac{g^2}{s}\int dz_*\bar\psi\not{p}_1\psi(z_*) \sim \frac{g}{s}\int dz_* D^iF_{\bullet i}(z_*) \sim l_\perp^2 U \sim l_\perp^2$ and each extra integration inside the shock wave brings extra $\sigma_*$. The first lines in r.h.s.'s of eqs. (A.34) and (A.35) are of order of $\frac{g^2p_i}{\alpha^2s^2}\int dz_*\bar\psi\not{p}_1\psi(z_*) \sim g\frac{p_il_\perp^2}{\alpha^2 s}\sigma_*$ so they can be neglected in comparison to the corresponding term $\frac{g}{\alpha s}\int dz_*\, F_{\bullet i}(z_*) \sim g\frac{l_i}{\alpha}$ in eq. (B.6). As to the terms (A.36) and (A.37), they are of the same odrer of magnitude as next-to-leading terms $\sim g_{\mu\nu}$ in eq. (B.6) so we keep them for now. With these approximations we obtain

$$\begin{aligned}\mathcal{Q}^{ab}_{\mu\nu}(x_*,y_*;p_\perp) &= -\frac{4ig}{\alpha^2s^3}p_{2\mu}p_{2\nu}\int_{y_*}^{x_*}dz_*([x_*,z_*]D^jF_{\bullet j}(z_*)[z_*,y_*])^{ab}\\&\quad+\frac{g^2}{\alpha s^2}\int_{y_*}^{x_*}dz_*\int_{y_*}^{z_*}dz'_*\bar\psi(z_*)[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\gamma_\mu^\perp\not{p}_1\gamma_\nu^\perp\psi(z'_*),\\ \bar{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp) &= -\frac{g^2}{\alpha s^2}\int_{y_*}^{x_*}dz_*\int_{z_*}^{x_*}dz'_*\bar\psi\gamma_\nu^\perp\not{p}_1\gamma_\mu^\perp[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\psi(z'_*)\end{aligned}\tag{B.7}$$

and the gluon propagator is given by eq. (A.45) with the above $\mathcal{G}_{\mu\nu}$, $\mathcal{Q}_{\mu\nu}$, and $\bar{\mathcal{Q}}_{\mu\nu}$:

$$\begin{aligned}&\langle A^a_\mu(x)A^b_\nu(y)\rangle\\&=\Big[-\theta(x_*-y_*)\int_0^\infty\frac{d\!\!{}^-\alpha}{2\alpha}+\theta(y_*-x_*)\int_{-\infty}^0\frac{d\!\!{}^-\alpha}{2\alpha}\Big]e^{-i\alpha(x-y)_\bullet}\big\{(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}\\&\quad\times\big[\mathcal{G}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)+\mathcal{Q}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)\big]|y_\perp)+(y_\perp|\bar{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}(x-y)_*}|x_\perp)\big\}\end{aligned}\tag{B.8}$$

As in the scalar case, it is easy to see that eq. (B.8) holds true if the point $y$ is inside the shock wave and the point $x$ anywhere.

Similarly, in the complex conjugate amplitude the gluon propagator is given by eq. (A.46) with

$$\begin{aligned}&\tilde{\mathcal{G}}_{\mu\nu}(x_*,y_*;p_\perp)\\&=g_{\mu\nu}[x_*,y_*]+g\int_{x_*}^{y_*}dz_*\;[x_*,z_*]\Big(\frac{2i}{\alpha s^2}(z-x)_*g_{\mu\nu}\{2\tilde F_{\bullet j}(z_*)[z_*,y_*]p^j+i\tilde D^j\tilde F_{\bullet j}(z_*)[z_*,y_*]\}\\&\quad-\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\tilde F_{\bullet j}(z_*)[z_*,y_*]\Big)\\&\quad+\frac{8g^2}{\alpha s^3}\int_{x_*}^{y_*}dz_*\int_{z_*}^{y_*}dz'_*[x_*,z_*]\Big(\big[-ig_{\mu\nu}(z-x)_*-\frac{2}{\alpha s}p_{2\mu}p_{2\nu}\big]\tilde F_{\bullet j}(z_*)[z_*,z'_*]\tilde F_\bullet{}^j(z'_*)[z'_*,y_*]\Big)\end{aligned}\tag{B.9}$$

and

$$\begin{aligned}\tilde{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp) &= \frac{4ig}{\alpha^2s^3}p_{2\mu}p_{2\nu}\int_{x_*}^{y_*}dz_*\big([x_*,z_*]\tilde D^j\tilde F_{\bullet j}(z_*)[z_*,y_*]\big)^{ab}\\&\quad+\frac{g^2}{\alpha s^2}\int_{x_*}^{y_*}dz_*\int_{z_*}^{y_*}dz'_*\bar{\tilde\psi}(z_*)\gamma_\mu^\perp\not{p}_1\gamma_\nu^\perp[z_*,x_*]t^a[x_*,y_*]t^b[y_*,z'_*]\tilde\psi(z'_*),\\ \bar{\tilde{\mathcal{Q}}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp) &= -\frac{g^2}{\alpha s^2}\int_{x_*}^{y_*}dz_*\int_{x_*}^{z_*}dz'_*\;\bar{\tilde\psi}\gamma_\nu^\perp\not{p}_1\gamma_\mu^\perp[z_*,y_*]t^b[y_*,x_*]t^a[x_*,z'_*]\tilde\psi(z'_*)\end{aligned}\tag{B.10}$$

The expressions (B.9) and (B.10) are valid for point $x$ inside the shock wave (and point $y$ inside or outside).

The corresponding expressions for the Lipatov vertex of gluon production are given by eqs. (A.47)–(A.50) with $\mathcal{G}$, $\mathcal{Q}$, and $\bar{\mathcal{Q}}$ changed accordingly.

## B.2 Propagators with both points outside the shock wave

In this section we will find the propagators with both points outside the shock wave. Again, we assume that the characteristic shock-wave transverse momenta are of order of transverse momenta of "quantum" fields with $\alpha > \sigma'$. As discussed in section 2, we consider the width of the shock-wave to be small but finite, consequently we can not recycle formulas from ref. [8, 9] for the infinitely thin shock-wave.

### B.2.1 Scalar propagator

As in the previous section, for simplicity we start with the scalar propagator (A.1)

$$(x|\frac{1}{P^2+i\epsilon}|y) \overset{x_*\geq y_*}{=} -i\int_0^\infty \frac{d\!\!{}^-\alpha}{2\alpha}\, e^{-i\alpha(x-y)_\bullet}(x_\perp|\mathrm{Pexp}\Big\{-i\int_{y_*}^{x_*} dz_*\Big[\frac{p_\perp^2}{\alpha s} - \frac{2g}{s}A_\bullet(z_*)\Big]\Big\}|y_\perp) \tag{B.11}$$

The Pexp in the r.h.s. of eq. (B.11) can be transformed to

$$\begin{aligned}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\mathrm{Pexp}\Big\{ig\int_{y_*}^{x_*} d\frac{2}{s}z_*\, e^{i\frac{p_\perp^2}{\alpha s}z_*}A_\bullet(z_*)e^{-i\frac{p_\perp^2}{\alpha s}z_*}\Big\}e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp) = \int d^2z_\perp d^2z'_\perp \\ \times(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)(z_\perp|\mathrm{Pexp}\Big\{ig\int_{y_*}^{x_*} d\frac{2}{s}z_*\; e^{i\frac{p_\perp^2}{\alpha s}z_*}A_\bullet(z_*)e^{-i\frac{p_\perp^2}{\alpha s}z_*}\Big\}|z'_\perp)(z'_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)\end{aligned} \tag{B.12}$$

Next, we use the expansion (B.2) at $y_* = 0$

$$e^{i\frac{p_\perp^2}{\alpha s}z_*}A_\bullet e^{-i\frac{p_\perp^2}{\alpha s}z_*} = A_\bullet - \frac{z_*}{\alpha s}(2p^iF_{\bullet i} - iD^iF_{\bullet i}) + \ldots \tag{B.13}$$

This is an expansion around the light cone $z_\perp + \frac{2}{s}z_*p_1$ with the parameter of the expansion $\sim \frac{p_\perp^2}{\alpha s}\sigma_* \ll 1$. Note that similarly to eq. (B.2) we need to keep the second term of this expansion since the first term forms gauge links.

From eqs. (A.4) and (A.5) we obtain (cf. eq. (A.6))

$$\begin{aligned}(x|\frac{1}{P^2+i\epsilon}|y) &= \Big[-i\theta(x_*-y_*)\int_0^\infty \frac{d\!\!{}^-\alpha}{2\alpha} + i\theta(y_*-x_*)\int_{-\infty}^0 \frac{d\!\!{}^-\alpha}{2\alpha}\Big]\, e^{-i\alpha(x-y)_\bullet} \\ &\quad\times\int d^2z_\perp(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\mathcal{O}(x_*,y_*;p_\perp)|z_\perp)(z_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)\end{aligned} \tag{B.14}$$

where

$$\begin{aligned}&\mathcal{O}(x_*,y_*;p_\perp) \\ &= [x_*,y_*] - \frac{2ig}{\alpha s^2}\int_{y_*}^{x_*} dz_*\; z_*\big(2p^j[x_*,z_*]F_{\bullet j}(z_*) - i[x_*,z_*]D^jF_{\bullet j}(z_*)\big)[z_*,y_*] \\ &\quad + \frac{8ig^2}{\alpha s^3}\int_{y_*}^{x_*} dz_*\int_{y_*}^{z_*} dz'_*\; z'_*[x_*,z_*]F_{\bullet j}(z_*)[z_*,z'_*]F_\bullet{}^j(z'_*)[z'_*,y_*] + \ldots\end{aligned} \tag{B.15}$$

Here the transverse arguments of all fields turn effectively to $z_\perp$. Note that this expression is equal to eq. (A.5) at $y_* = 0$. For the complex conjugate amplitude one obtains (cf. eq. (A.12))

$$(x|\frac{1}{P^2 - i\epsilon}|y) = \left[i\theta(y_* - x_*)\int_0^\infty \frac{\bar{d}\alpha}{2\alpha} - i\theta(x_* - y_*)\int_{-\infty}^0 \frac{\bar{d}\alpha}{2\alpha}\right] e^{-i\alpha(x-y)_\bullet} \times \int d^2z_\perp (x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\tilde{\mathcal{O}}(x_*, y_*; p_\perp)|z_\perp)(z_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp) \tag{B.16}$$

where

$$\tilde{\mathcal{O}}(x_*, y_*; p_\perp) = [x_*, y_*] + \frac{2ig}{\alpha s^2}\int_{x_*}^{y_*} dz_*\ z_*[x_*, z_*](2\tilde{F}_{\bullet j}(z_*)[z_*, y_*]p^j + i\tilde{D}^j\tilde{F}_{\bullet j}(z_*)[z_*, y_*]) - \frac{8ig^2}{\alpha s^3}\int_{x_*}^{y_*} dz_*\int_{z_*}^{y_*} dz'_*\ z_*[x_*, z_*]\tilde{F}_{\bullet j}(z_*)[z_*, z'_*]_z\tilde{F}_\bullet{}^j(z'_*)[z'_*, y_*] \tag{B.17}$$

Again, this expression can be obtained from eq. (A.11) by taking $x_* \to 0$ in parentheses.

### B.2.2 The emission vertex

Similarly to eq. (A.14) we get

$$\begin{aligned}\lim_{k^2\to 0} k^2(k|\frac{1}{P^2 + i\epsilon}|y) &= \int dx_* d^2x_\perp\ e^{i\frac{k_\perp^2}{\alpha s}x_* - i(k,x)_\perp} \\ &\times\left[\frac{\partial}{\partial x_*} - \frac{i}{\alpha s}\partial_{x_\perp}^2\right]\theta(x-y)_*\int d^2z_\perp\ (x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)(z_\perp|\mathcal{O}(x_*, y_*; p_\perp)e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)e^{i\alpha y_\bullet} \\ &= \int dx_*\frac{\partial}{\partial x_*}\theta(x-y)_*(k_\perp|\mathcal{O}(x_*, y_*; p_\perp)e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)e^{i\alpha y_\bullet} \\ &= \int d^2z_\perp\ e^{-i(k,z)_\perp}\mathcal{O}(\infty, y_*; z_\perp; k)(z_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)e^{i\alpha y_\bullet}\end{aligned} \tag{B.18}$$

where $\mathcal{O}(\infty, y_*, z_\perp; k)$ is obtained from eq. (B.15)

$$\begin{aligned}\mathcal{O}(\infty, y_*, z_\perp; k) &\equiv e^{i(k,z)_\perp}(k_\perp|\mathcal{O}(\infty, y_*; p_\perp)|z_\perp) \\ &= [\infty, y_*]_z - \frac{2ig}{\alpha s^2}\int_{y_*}^\infty dz_*\ z_*[\infty, z_*]_z(2k^jF_{\bullet j} - iD^jF_{\bullet j})(z_*, z_\perp)[z_*, y_*]_z \\ &\quad + \frac{8ig^2}{\alpha s^3}\int_{y_*}^\infty dz_*\int_{y_*}^{z_*} dz'_*\ z'_*[\infty, z_*]_zF_{\bullet j}(z_*, z_\perp)[z_*, z'_*]_zF_\bullet{}^j(z'_*, z_\perp)[z'_*, y_*]_z\end{aligned} \tag{B.19}$$

For the complex conjugate amplitude we get

$$\begin{aligned}\lim_{k^2\to 0} k^2(x|\frac{1}{P^2 - i\epsilon}|k) &= \int dy_* d^2y_\perp\ e^{-i\frac{k_\perp^2}{\alpha s}y_* + i(k,y)_\perp} \\ &\times\left[\frac{\partial}{\partial y_*} + \frac{i}{\alpha s}\partial_{y_\perp}^2\right]\theta(y-x)_*\int d^2z_\perp\ (x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)(z_\perp|\tilde{\mathcal{O}}(x_*, y_*; p_\perp)e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)e^{-i\alpha x_\bullet} \\ &= \int dy_*\frac{\partial}{\partial y_*}\theta(y-x)_*(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\tilde{\mathcal{O}}(x_*, y_*; p_\perp)|k_\perp)e^{-i\alpha x_\bullet} \\ &= \int d^2z_\perp\ e^{i(k,z)_\perp}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)\tilde{\mathcal{O}}(x_*, \infty; z_\perp; k)e^{-i\alpha x_\bullet}\end{aligned} \tag{B.20}$$

where $\tilde{\mathcal{O}}(\infty, y_*, z_\perp; k)$ is obtained from eq. (B.17) in a usual way

$$\tilde{\mathcal{O}}(x_*, \infty; z_\perp; k) \equiv e^{-i(k,z)_\perp}(z_\perp|\tilde{\mathcal{O}}(x_*, \infty; p_\perp)|k_\perp) \tag{B.21}$$
$$= [x_*, \infty]_z + \frac{2ig}{\alpha s^2}\int_{x_*}^{\infty} dz_*\ z_*[x_*, z_*]_z(2k^j\tilde{F}_{\bullet j} + i\tilde{D}^j\tilde{F}_{\bullet j})(z_*, z_\perp)[z_*, \infty]_z$$
$$-\frac{8ig^2}{\alpha s^3}\int_{x_*}^{\infty} dz_* \int_{z_*}^{\infty} dz'_*\ z_*[x_*, z_*]_z\tilde{F}_{\bullet j}(z_*, z_\perp)[z_*, z'_*]_z\tilde{F}_\bullet{}^j(z'_*, z_\perp)[z'_*, \infty]_z$$

### B.2.3 Gluon propagator in the shock-wave background

The gluon propagator in a background gluon field (A.18) can be rewritten as

$$i\langle A^a_\mu(x)A^b_\nu(y)\rangle = (x|\frac{1}{P^2 + 2igF + i\epsilon}|y)^{ab}_{\mu\nu} \stackrel{x_*\geq y_*}{=} -i\int_0^\infty \frac{d\!\!\bar{}\,\alpha}{2\alpha}\ e^{-i\alpha(x-y)_\bullet} \tag{B.22}$$
$$\times(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\mathrm{Pexp}\Big\{\frac{2ig}{s}\int_{y_*}^{x_*} dz_*\ e^{i\frac{p_\perp^2}{\alpha s}z_*}\Big(A_\bullet + \frac{i}{\alpha}F\Big)(z_*)e^{-i\frac{p_\perp^2}{\alpha s}z_*}\Big\}_{\mu\nu} e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)^{ab}$$

Using the expansion (A.19) at $y_* = 0$ we obtain with our accuracy

$$e^{i\frac{p_\perp^2}{\alpha s}z_*}\Big(A_\bullet g_{\mu\nu} + \frac{i}{\alpha}F_{\mu\nu}\Big)e^{-i\frac{p_\perp^2}{\alpha s}z_*} \tag{B.23}$$
$$= g_{\mu\nu}\Big[A_\bullet - \frac{z_*}{\alpha s}(2p^jF_{\bullet j} - iD^jF_{\bullet j})\Big] + \frac{i}{\alpha}F_{\mu\nu} + i\frac{z_*}{\alpha^2 s}\Big(2p^jD_jF_{\mu\nu} - iD^jD_jF_{\mu\nu}\Big) + \ldots$$

Similarly to eq. (A.21) we get

$$(x|\frac{1}{P^2 + 2igF + i\epsilon}|y)^{ab}_{\mu\nu} = \Big[-i\theta(x_* - y_*)\int_0^\infty \frac{d\!\!\bar{}\,\alpha}{2\alpha} + i\theta(y_* - x_*)\int_{-\infty}^0 \frac{d\!\!\bar{}\,\alpha}{2\alpha}\Big]\ e^{-i\alpha(x-y)_\bullet}$$
$$\times\int d^2z_\perp(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)(z_\perp|\mathcal{G}^{ab}_{\mu\nu}(x_*, y_*; p_\perp)e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp) \tag{B.24}$$

where

$$\mathcal{G}_{\mu\nu}(x_*, y_*; p_\perp) \tag{B.25}$$
$$= g_{\mu\nu}[x_*, y_*] + g\int_{y_*}^{x_*} dz_*\ \Big(-\frac{2i}{\alpha s^2}z_*g_{\mu\nu}(2p^j[x_*, z_*]F_{\bullet j}(z_*) - i[x_*, z_*]D^jF_{\bullet j}(z_*))$$
$$+\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu} - \delta^j_\nu p_{2\mu})\Big\{[x_*, z_*]F_{\bullet j}(z_*) + \frac{2iz_*}{\alpha s}p^k[x_*, z_*]D_kF_{\bullet j}(z_*)\Big\}\Big)[z_*, y_*]$$
$$+\frac{8g^2}{\alpha s^3}\int_{y_*}^{x_*} dz_* \int_{y_*}^{z_*} dz'_*\ \Big[ig_{\mu\nu}z'_* - \frac{2}{\alpha s}p_{2\mu}p_{2\nu}\Big][x_*, z_*]F_{\bullet j}(z_*)[z_*, z'_*]F_\bullet{}^j(z'_*)[z'_*, y_*] + \ldots$$

Let us consider now quark terms coming from figure 7. From eq. (A.24) it is clear that this contribution can be parameterized similarly to eq. (A.31):

$$\langle A^a_\mu(x)A^b_\nu(y)\rangle_{\text{figure 7}} \tag{B.26}$$
$$= \Big[-\theta(x_* - y_*)\int_0^\infty \frac{d\!\!\bar{}\,\alpha}{2\alpha} + \theta(y_* - x_*)\int_{-\infty}^0 \frac{d\!\!\bar{}\,\alpha}{2\alpha}\Big]\ e^{-i\alpha(x-y)_\bullet}\int d^2z_\perp\ \Big[(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)$$
$$\times(z_\perp|\mathcal{Q}^{ab}_{\mu\nu}(x_*, y_*; p_\perp)e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp) + (y_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|z_\perp)(z_\perp|\bar{\mathcal{Q}}^{ab}_{\mu\nu}(x_*, y_*; p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}x_*}|x_\perp)\Big]$$

where $\mathcal{Q}^{ab}_{\mu\nu}$ and $\mathcal{Q}^{ab}_{\mu\nu}$ are given by expressions (A.32)–(A.37) with $z_*-y_*\to z_*$ (and similarly $z'_*-y_*\to z'_*$). With our accuracy only the first terms in these expressions survive so $\mathcal{Q}^{ab}_{\mu\nu}$ and $\bar{\mathcal{Q}}^{ab}_{\mu\nu}$ are given by eq. (B.7) from previous section.

Adding gluon contribution (B.24) one obtains the final expression for gluon propagator in a shock-wave background:

$$\begin{aligned}\langle A^a_\mu(x)A^b_\nu(y)\rangle=&\Big[-\theta(x_*-y_*)\int_0^\infty\frac{d\!\!\!^-\alpha}{2\alpha}+\theta(y_*-x_*)\int_{-\infty}^0\frac{d\!\!\!^-\alpha}{2\alpha}\Big]\,e^{-i\alpha(x-y)_\bullet}\\&\times\int d^2z_\perp\Big[(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)(z_\perp|[\mathcal{G}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)+\mathcal{Q}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)]e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)\\&+(y_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|z_\perp)(z_\perp|\bar{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}x_*}|x_\perp)\Big]\end{aligned}\tag{B.27}$$

where $\mathcal{G}^{ab}_{\mu\nu}$ is given by eq. (B.25) and $\mathcal{Q}^{ab}_{\mu\nu}$, $\bar{\mathcal{Q}}^{ab}_{\mu\nu}$ are given by eq. (B.7).

Similarly, for the complex conjugate amplitude one obtains (cf. eq. (A.46))

$$\begin{aligned}\langle\tilde{A}^a_\mu(x)\tilde{A}^b_\nu(y)\rangle=&\Big[-\theta(y_*-x_*)\int_0^\infty\frac{d\!\!\!^-\alpha}{2\alpha}+\theta(x_*-y_*)\int_{-\infty}^0\frac{d\!\!\!^-\alpha}{2\alpha}\Big]e^{-i\alpha(x-y)_\bullet}\\&\times\int d^2z_\perp\Big[(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}\,[\tilde{\mathcal{G}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)+\tilde{\mathcal{Q}}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)]|z_\perp)(z_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)\\&+(y_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|z_\perp)(z_\perp|\bar{\tilde{\mathcal{Q}}}{}^{ab}_{\mu\nu}(x_*,y_*;p_\perp)e^{-i\frac{p_\perp^2}{\alpha s}x_*}|x_\perp)\Big]\end{aligned}\tag{B.28}$$

where

$$\begin{aligned}&\tilde{\mathcal{G}}_{\mu\nu}(x_*,y_*;p_\perp)\\&=g_{\mu\nu}[x_*,y_*]+g\int_{x_*}^{y_*}dz_*\ [x_*,z_*]\Big\{\frac{2iz_*}{\alpha s^2}g_{\mu\nu}(2\tilde{F}_{\bullet j}(z_*)[z_*,y_*]p^j+i\tilde{D}^j\tilde{F}_{\bullet j}[z_*,y_*])\\&\quad-\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu}-\delta^j_\nu p_{2\mu})\big(\tilde{F}_{\bullet j}(z_*)[z_*,y_*]+2i\frac{z_*}{\alpha s}\tilde{D}_l\tilde{F}_{\bullet j}(z_*)[z_*,y_*]k^l\big)\Big\}\\&\quad+\frac{8g^2}{\alpha^2s^3}\int_{x_*}^{y_*}dz_*\int_{z_*}^{y_*}dz'_*\ [x_*,z_*]\Big[-i\alpha g_{\mu\nu}z_*-\frac{2}{s}p_{2\mu}p_{2\nu}\Big]\ \tilde{F}_{\bullet j}(z_*)[z_*,z'_*]\tilde{F}_\bullet{}^j(z'_*)[z'_*,y_*]\end{aligned}\tag{B.29}$$

and $\tilde{\mathcal{Q}}^{ab}_{\mu\nu}$, $\bar{\tilde{\mathcal{Q}}}{}^{ab}_{\mu\nu}$ are given by eq. (B.10). Note that the transverse coordinates of all fields are effectively $z_\perp$.

### B.2.4 Gluon emission vertex

Similarly to eq. (B.18) one obtains from eq. (B.27)

$$\lim_{k^2\to0}k^2i\langle A^a_\mu(k)A^b_\nu(y)\rangle=\int d^2z_\perp\ e^{-i(k,z)_\perp}\mathcal{O}^{ab}_{\mu\nu}(\infty,y_*,z_\perp;k)(z_\perp|e^{i\frac{p_\perp^2}{\alpha s}y_*}|y_\perp)e^{i\alpha y_\bullet}\tag{B.30}$$

where $\mathcal{O}^{ab}_{\mu\nu}(\infty, y_*, z_\perp; k)$ is given by eqs. (A.47)–(A.48). With our accuracy we get

$$\begin{aligned}
&\mathcal{O}^{ab}_{\mu\nu}(\infty, y_*, z_\perp; k) = g_{\mu\nu}[\infty, y_*]^{ab}_z \\
&+g\int_{y_*}^{\infty} dz_* \Big([\infty, z_*]_z\Big[-\frac{2iz_*}{\alpha s^2}g_{\mu\nu}(2k^j - iD^j) + \frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu} - \delta^j_\nu p_{2\mu})\Big]F_{\bullet j}(z_*, z_\perp)[z_*, y_*]_z\Big)^{ab} \\
&+\frac{4g}{\alpha^2 s^3}\int_{y_*}^{\infty} dz_* \Big\{ - ip_{2\mu}p_{2\nu}[\infty, z_*]_z D^j F_{\bullet j}(z_*, z_\perp)[z_*, y_*]_z \\
&+g\int_{y_*}^{z_*} dz'_* \Big[2i\alpha g_{\mu\nu}z'_* - \frac{4}{s}p_{2\mu}p_{2\nu}\Big]\,[\infty, z_*]_z F_{\bullet j}(z_*, z_\perp)[z_*, z'_*]_z F_\bullet{}^j(z'_*, z_\perp)[z'_*, y_*]_z\Big\}^{ab} \\
&+\frac{g^2}{\alpha s^2}\Big\{\int_{y_*}^{\infty} dz_* \int_{y_*}^{z_*} dz'_* \bar\psi(z_*, z_\perp)[z_*, \infty]_z t^a[\infty, y_*]_z t^b[y_*, z'_*]_z \gamma^\perp_\mu \not{p}_1 \gamma^\perp_\nu \psi(z'_*, z_\perp) \\
&-\int_{y_*}^{\infty} dz_* \int_{z_*}^{\infty} dz'_* \bar\psi(z_*, z_\perp)\gamma^\perp_\nu \not{p}_1 \gamma^\perp_\mu [z_*, y_*]_z t^b[y_*, \infty]_z t^a[\infty, z'_*]_z \psi(z'_*, z_\perp)\Big\}
\end{aligned} \tag{B.31}$$

For the gluon emission in the complex conjugate amplitude one obtains (cf. eq. (B.20)

$$-\lim_{k^2\to 0} k^2 i\langle \tilde A^a_\mu(x)\tilde A^b_\nu(k)\rangle \;= \int d^2z_\perp\; e^{i(k,z)_\perp}(x_\perp|e^{-i\frac{p_\perp^2}{\alpha s}x_*}|z_\perp)\tilde{\mathcal{O}}^{ab}_{\mu\nu}(x_*, \infty, z_\perp; k)e^{-i\alpha x_\bullet} \tag{B.32}$$

where $\tilde{\mathcal{O}}^{ab}_{\mu\nu}(\infty, y_*, z_\perp; k)$ is given by eqs. (A.49)–(A.50). With our accuracy

$$\begin{aligned}
&\tilde{\mathcal{O}}^{ab}_{\mu\nu}(x_*, \infty, z_\perp; k) = g_{\mu\nu}[x_*, \infty]^{ab}_z \\
&+g\int_{x_*}^{\infty} dz_* \;\Big([x_*, z_*]_z\Big\{\frac{2iz_*}{\alpha s^2}g_{\mu\nu}(2k^j + i\tilde D^j)\tilde F_{\bullet j}(z_*, z_\perp)[z_*, \infty]_z \\
&-\frac{4}{\alpha s^2}(\delta^j_\mu p_{2\nu} - \delta^j_\nu p_{2\mu})\Big(1 + 2ik^l\frac{z_*}{\alpha s}\tilde D_l\Big)\tilde F_{\bullet j}(z_*, z_\perp)[z_*, \infty]_z\Big\}\Big)^{ab} \\
&+\frac{4g}{\alpha^2 s^3}\int_{x_*}^{\infty} dz_* \;\Big([x_*, z_*]_z\Big\{ip_{2\mu}p_{2\nu}\tilde D^j\tilde F_{\bullet j}(z_*, z_\perp)[z_*, \infty]_z \\
&+g\int_{z_*}^{\infty} dz'_*\Big[-2i\alpha g_{\mu\nu}z_* - \frac{4}{s}p_{2\mu}p_{2\nu}\Big]\tilde F_{\bullet j}(z_*, z_\perp)[z_*, z'_*]_z\tilde F_\bullet{}^j(z'_*, z_\perp)[z'_*, \infty]_z\Big\}\Big)^{ab} \\
&+\frac{g^2}{\alpha s^2}\Big\{\int_{x_*}^{\infty} dz_* \int_{z_*}^{\infty} dz'_* \tilde{\bar\psi}(z_*, z_\perp)[z_*, x_*]_z t^a[x_*, \infty]_z t^b[\infty, z'_*]_z \gamma^\perp_\mu \not{p}_1 \gamma^\perp_\nu \tilde\psi(z'_*, z_\perp) \\
&-\int_{x_*}^{\infty} dz_* \int_{x_*}^{z_*} dz'_* \tilde{\bar\psi}(z_*, z_\perp)\gamma^\perp_\nu \not{p}_1 \gamma^\perp_\mu [z_*, \infty]_z t^b[\infty, x_*]_z t^a[x_*, z'_*]_z \tilde\psi(z'_*, z_\perp)\Big\}.
\end{aligned} \tag{B.33}$$



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